Volume 10, September 2011

ISSN 1815-3852

Journal of the Association of Arab Universities for Basic and Applied Sciences

مجلة اتحاد الجامعات العربية للعلوم الأساسية والتطبيقية

Editor-in-Chief
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A Biannual Scientific Refereed Journal, Published by the College of Science, University of Bahrain.

Indexed internationally by Ulrich's database.

Journal of the Association of Arab Universities for Basic and Applied Sciences (JAAUBAS)

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ISSN 1815-3852

Journal of the Association of Arab Universities for Basic and Applied Sciences

Volume 10, September 2011

Published by

University of Bahrain

University of Bahrain

Journal of the Association of Arab Universities for Basic and Applied Sciences

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Foreword

It gives me great pleasure to announce to researchers in the University of Bahrain and in all Arab Universities that JAAUBAS is now endorsed among Elsevier and B.V. Publication. Your contribution to JAAUBAS can now be submitted directly through the journal website using http://www.ees.elsevier.com/jaaubas.

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Dr Haifa Al Maskati
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تجمعات الكائنات القاعية في بيئات الأعشاب البحرية في البحرين

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الملخص:

تساهم بيئات الأعشاب البحرية في البحرين بشكل كبير في إنتاجية الأسماك المحلية وتوفير مصادر الغذاء للأنواع المعرضة للخطر. بيد أن هذه النظم البيئية تحت التهديدات المستمرة نتيجة الضغوط البشرية، بما في ذلك الردم والتجريف و النفايات الصناعية السائلة، ومياه المجاري، والمياه المالحة من محطات التحلية. مسح بيئات الأعشاب البحرية وما يرتبط بها من الكائنات القاعية يساهم في الحفاظ على هذه النظم البيئية الحساسة. أخذت عينات من تجمعات الكائنات القاعية من بيئات الأعشاب البحرية من موقعين، غرب و شرق البحرين والتي تتعرض لظروف بيئية مختلفة. توصلت الدراسة إلى وجود اختلافات في تركيبة تجمعات الكائنات القاعية بين الموقعين. حيث كانت القشريات تهيمن عدديا في الموقع الغربي، في حين كانت الرخويات المجموعة المهيمنة في الموقع الشرقي. الملوحة والرواسب أهم العوامل البيئية الرئيسية المسؤولة عن تفسير 44 ٪ من أنماط تجمعات الكائنات في منطقة الدراسة. هذا وقد غطت الأعشاب البحرية 3.6% ± 95 و % ± 78 7.4من الموقعين الغربي والشرقي على التوالي. الآثار المترتبة على هذه الدراسة تتيح إحراز قرارات أفضل بشأن الحفاظ على نظم بيئات الأعشاب البحرية في البحرين.

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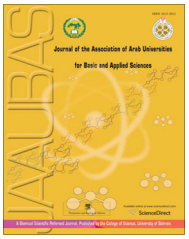

ORIGINAL ARTICLE

Assemblages of macro-fauna associated with two seagrass beds in Kingdom of Bahrain: Implications for conservation

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Available online 21 July 2011

KEYWORDS

Seagrass conservation;
Macrobenthos;
Salinity;
Coastal developments;
Bahrain

Abstract Seagrass ecosystems in Bahrain contribute significantly to the productivity of local fisheries and provide food sources and nursery grounds for vulnerable species. However, these ecosystems are under continuous threats from anthropogenic pressures, including reclamation and dredging, industrial effluents, domestic discharges, and brine water from desalination plants. Surveying seagrass beds and associated macro-fauna is required to contribute to management and conservation effort of these sensitive ecosystems. Macrobenthic assemblages were sampled from two seagrass beds off the western and eastern coasts of Bahrain that are subjected to different environmental conditions. Differences in structure and composition among the assemblages between the two sampling sites were detected. The western site was numerically dominated by crustaceans, while molluscs were the dominant group in the eastern site. Salinity and sediments were the main environmental factors responsible for explaining 44% of the community patterns in the study areas. Seagrass cover was 95 ± 3.6% and 78 ± 7.4% for the western and eastern sites, respectively. Implications of this study may allow better decisions to be made concerning the conservation of seagrass ecosystems in Bahrain.

1. Introduction

Seagrass meadows are highly productive ecosystems that provide important ecological functions and economic services (Sheppard et al., 1992; Duarte, 2002; Duffy, 2006). Ecologically, seagrass ecosystems provide food sources and function as nursery grounds for threatened species such as turtles and dugongs (Price et al., 1993; Preen, 2004). They can also improve water quality by stabilizing loose sediment and by filtering some pollutants out of the water (Duffy, 2006). Economically, they contribute significantly to the productivity of local fisheries (Vousden, 1995; Abdulqader, 1999).

Despite the important of these habitats, seagrass ecosystems are presently experiencing global decline primarily because of

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Peer review under responsibility of University of Bahrain.
doi:10.1016/j.jaubas.2011.06.004

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human disturbances, such as direct destruction of physical habitat, interference with the biological functions associated with seagrass beds, overexploitation, eutrophication and pollution (Erftemeijer and Lewis, 2006; Airoldi and Beck, 2007; Burkholder et al., 2007; Cabaco et al., 2008). In some parts of the world such as Bahrain, coastal and marine environments are the prime target for most of the major housing, recreational, economic and industrial developments (Naser et al., 2008). Reclamation and dredging activities may contribute directly or indirectly to the loss of seagrass beds, due to direct physical removal and burial, and the increase in turbidity levels (Zainal et al., 1993). Additionally, several land-based activities in the region affect the quality of coastal and marine environments, including industrial effluents, domestic discharges and brine water discharged by desalination plants. These activities are continuing threats for seagrass ecosystems in Bahrain. Therefore, there is a concern that the productivity of Bahraini coastal and marine environments could be affected by reduced functioning of seagrass ecosystems due to human-induced pressures (Abdulqader, 2001).

While efforts are being made to conserve seagrass ecosystems in Bahrain by establishing marine protected areas (Al-Zayani, 2003), there is still relatively little understanding of the role of seagrass habitat for associated plant, invertebrate, and fish communities. Therefore, baseline surveys of seagrass beds and associated macro-faunal assemblages are necessary to manage and conserve these fragile ecosystems (Naser, 2010). This study aims to characterize the community structure of macrobenthic assemblages associated with two seagrass beds located off the western and eastern coasts of Bahrain, and to explore the implications of conserving such beds in the light of the increased human pressure on the coastal and marine environments.

2. Materials and methods

2.1. Study area and sites selection

Seagrass beds are distributed along the southeast coast, and along the west coast of Bahrain (Vousden, 1995). Seagrasses in Bahrain are represented by three species, *Halodule uninervis*, *Halophila stipulacea* and *Halophila ovalis* (Phillips, 2003). Two subtidal seagrass beds were selected off the west (N 26°05.564, E 50°26.528) and east (N 26°00.840, E 50°43.124) coastlines of Bahrain (Fig. 1). The selected sites were of relatively comparable depths with approximate averages of 5 and 4 m for the western and eastern sites, respectively. At each site, a 1 km^2 sampling area was established and divided into nine stations at the intercepts points with a distance of 500 m between each one. This grid sampling design was adopted in order to investigate the spatial distribution of macrobenthic assemblages. The sampling was conducted in August 2006.

2.2. Measurement of environmental parameters

Environmental parameters including depth (*m*), water temperature (°C), salinity (psu), pH, dissolved oxygen (mg l^{-1}) and water transparency (*m*) were measured at each station using GARMIN FishFinder 240, glass thermometer, refractometer (Atago F/mill8901), Radiometer model pH 82, dissolve oxygen meter (Eil 7130) and Secchi disc, respectively. Measurements were taken at a depth of approximately 1 m below water surface.

2.3. Percentage cover of seagrass

Quadrats were photographed to estimate percentage cover seagrass non-destructively (Montefalcone, 2009). A quadrat (1 m^2) was deployed in each station by SCUBA diving and still photographs were taken. Percentage cover of seagrass at each station was calculated from five photo-quadrats using Image J 1.41 software. The mean percentage cover for the two sites was subsequently determined.

2.4. Sampling and treatment of macrobenthic invertebrates

Sediment samples were collected using hand-operated Van Veen grab (0.0675 m^2). In each station, four grabs were collected; three replicates for macroinvertebrates and one grab for organic content and grain size analysis of sediment. Macroinvertebrate samples were sieved in situ through a 1 mm sieve using seawater, transferred into labelled polyethylene bags, and stored under ice. At the laboratory, faunal samples were fixed using buffered formalin (4%) stained with Rose Bengal, and subsequently preserved using 70% ethanol. Organisms were sorted according to their taxonomic groups, counted and identified to the lowest possible taxonomic level using relevant identification guides (Jones, 1986; Green, 1994; Bosch et al., 1995; Richmond, 2002; Wehe and Fiege, 2002).

2.5. Sediment grain size analysis

The analysis of sediment particle size was carried out following Holme and McIntyre (1984), and involved sieving 50 g of homogenized sediment through a series of sieves on a mechanical sieve shaker (KARL KOLB). The weight of sediment fraction retained in each sieve was obtained and median particle size was determined. Another subsample of the homogenized sediment was used to determine the organic content by incinerating a known weight at a temperature of 550 °C for 12 h.

2.6. Statistical analysis

Univariate and multivariate analyses were employed to test for differences between the two sites. Ecological indices such as diversity index of Shannon–Wiener, Margalef's index of richness and Pielou's index of evenness were calculated. The spatial variation of macrobenthic assemblages was analysed by multidimensional scaling (MDS) based on Bray–Curtis dissimilarity index using square root transformed data. Environmental variables were analyzed based on Euclidean distance similarity measure. Environmental variables best correlated with patterns of macrobenthic assemblages were identified using Spearman coefficient (BIO-ENV analysis). Tests were performed using PRIMER® v 6 (Clarke and Gorley, 2006), and differences between the two sites were tested by one-way ANOVA using MINTAB v 12 statistical packages.

3. Results

3.1. Environmental parameters

There was a significant difference in salinity levels between western and eastern sites (55.1 ± 0.2 and 45.0 ± 0.0 psu for western and eastern sites, respectively ($P \leqslant 0.001$). Relatively

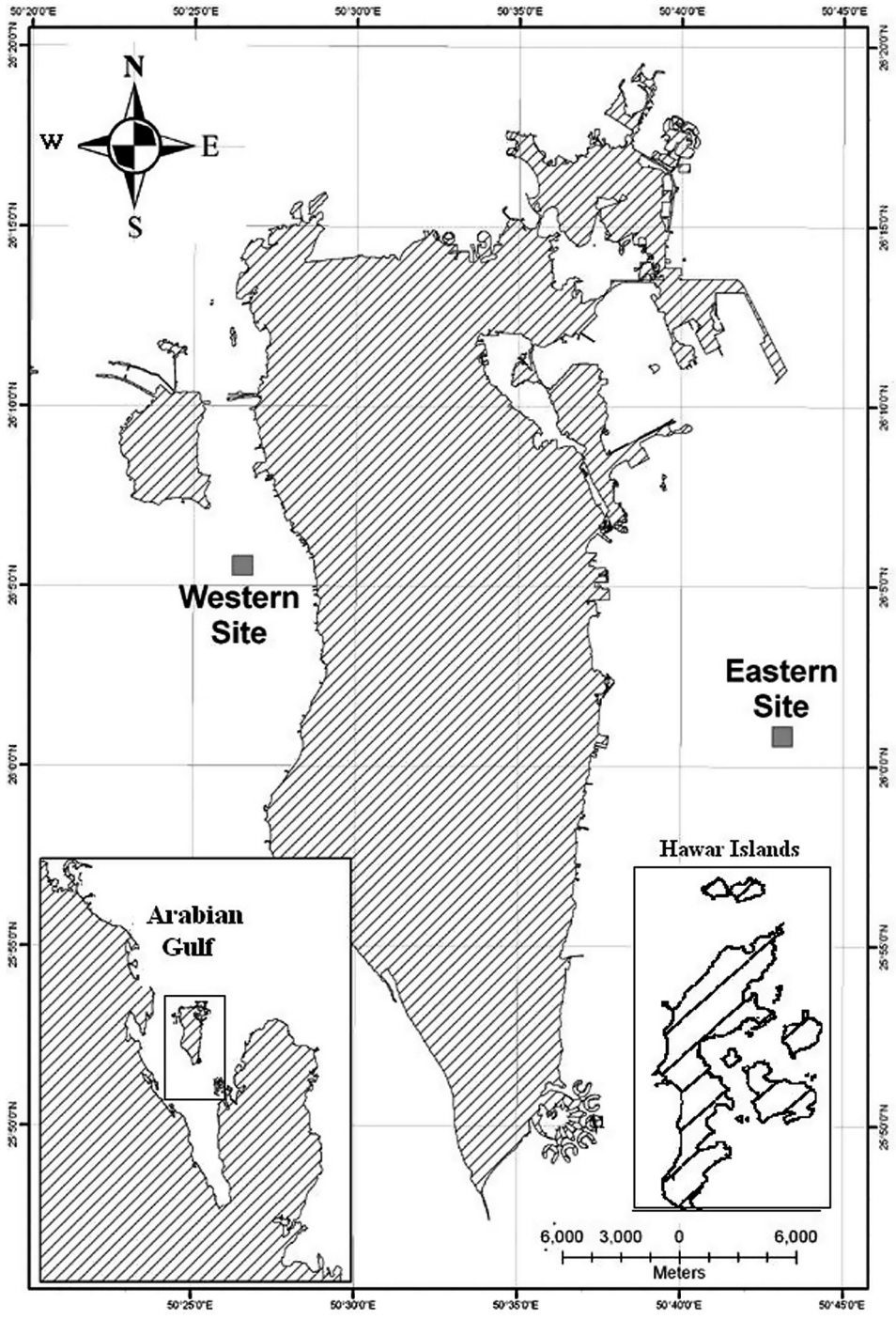

Figure 1 Map of Bahrain showing the locations of sampling sites.

comparable depths were detected in both selected seagrass beds (5.3 ± 1.9 and 3.9 ± 1.0 m for western and eastern sites, respectively). Sediments on both sites were categorized as medium sand. Although no significant difference was detected, sediment of western site was relatively finer ($Ø$ = 1.47 ± 0.7) than the eastern one ($Ø$ = 1.13 ± 0.4) (Table 1).

Non-parametric multi-dimensional scaling (MDS) (Fig. 2) for the environmental parameters revealed that eastern stations were clustering together indicating a higher level of similarity between these stations. Conversely, western stations were relatively scattered on the MDS. In particular, stations W1 and W2 showed higher levels of dissimilarity with the rest of stations, which could be attributed to the high percentage of clay (25%), and the depth (7.2 m), respectively.

3.2. Percentage cover of seagrass

Percentage cover of seagrass was higher in the western site (95 ± 3.6%) than the eastern one (78 ± 7.4%). The three species of seagrass were recorded in all stations of both sites with

Table 1 Environmental factors of the sampled station in the western and eastern sites.

Environmental variables	Depth (*m*)	Transparency (*m*)	Dissolved oxygen ($mg\,l^{-1}$)	pH	Salinity (psu)	Organic content (%)	Mean sediment particle (Ø)
Western site	5.3 ± 1.9	4.5 ± 1.2	5.0 ± 0.2	7.9 ± 0.3	55.1 ± 0.6	9.1 ± 3.8	1.47 ± 0.7
Eastern site	3.9 ± 1.0	3.4 ± 0.9	4.9 ± 0.3	7.9 ± 0.2	45.3 ± 0.0	2.9 ± 1.2	1.13 ± 0.4

Figure 2 MDS plot for square-root normalized environmental parameters using Euclidean distance. ANOSIM: $R = 0.514$, $P = 0.001$. E = eastern site, W = western site, and numbers correspond with number of stations.

H. stipulacea being the most abundant followed by *H. uninervis* and *H. ovalis*. Generally, seagrasses in the western site were taller and denser with more limited sand patches between them than those found in the eastern site.

3.3. Faunal community structure

Differences in community structure were detected between the western and eastern sites. A total of 519 individual organisms belonging to 42 species were recorded in the western site compared with 887 individuals belonging to 40 species in the eastern site. In the western site, crustaceans were the most abundant among the major taxonomic groups followed by polychaetes, molluscs, and the remaining groups (ascidians, sponges and cnidarians), respectively. Conversely, crustaceans were the least dominant group in the eastern site, which was dominated by molluscs followed by polychaetes and the remaining taxonomic groups (Fig. 3).

Ecological indices namely, Shannon-Wiener diversity (H′), species richness (Margalef's R), and evenness (Pielou's J) were relatively higher in the western site than the eastern one (Fig. 4). However, only the evenness index was statistically significantly different ($P = 0.045$).

The MDS of faunal abundance revealed a clear separation between samples collected from the western and eastern sites (Fig. 5) suggesting pronounced differences in their community structure (ANOSIM: $R = 0.541$, $P = 0.001$). A similarity of 35% separated the eastern from the western stations. However, stations W3 and W9 exhibited distinctive dissimilarities with the rest of stations. Station W3, the shallowest (2 m), was devoid of echinoderms while station W9 was distinguished by the highest and lowest numbers of crustaceans and molluscs, respectively.

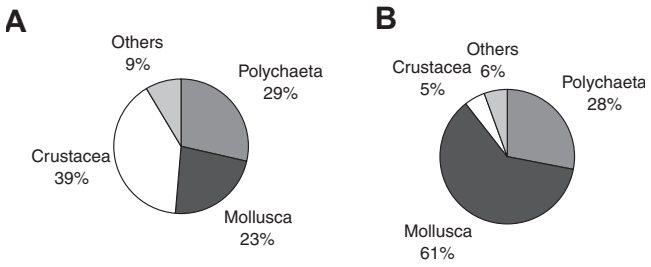

Figure 3 Numerical dominance among major taxonomic groups recorded in the (A) western and (B) eastern sites.

Correlations (Spearman) between environmental parameters and biotic assemblages indicated that 44% of the biota patterns were explained by a combination of salinity and the percentage of medium sand.

4. Discussion

Seagrass habitats support greater macro-fauna species diversity, abundance and biomass than adjacent unvegetated habitats (Coles and McCain, 1990; Ansari et al., 1991; Al-Khayat, 2007). But, there is often considerable variability in the macro-faunal assemblages associated with these seagrass beds (Worthington et al., 1992; Hemminga and Duarte, 2000), suggesting that various physical and chemical factors influence macro-faunal abundance and distribution within these seagrass meadows (Borum et al., 2004; Marba et al., 2006).

Salinity has a profound effect on seagrass distribution as well as abundance and composition of invertebrate fauna (Joyce et al., 2005). Despite the limited marine area of Bahrain, there are variations in salinity regimes around these islands. Salinities on the west coast of Bahrain are usually higher than those on the east coast (means of 50–57 and 43–45 psu for the west and east coasts, respectively) (Price et al., 1985). In this study, despite the high salinity in the western coast of Bahrain, seagrass cover was relatively high in the selected site. Seagrass cover is controlled by several interactive factors, including light, substratum type and water movements (Jones et al., 2002). The western site was characterized by high water visibility, fine sediment and moderate water movements in comparison with the eastern site. Price and Coles (1992) indicated that seagrass cover in the western Arabian Gulf coast shows significant positive correlation with latitude, but not with salinity, temperature or depth.

Sediment characteristics are major factors in governing the spatial distribution of marine benthos (Levinton, 2008). In the Arabian Gulf, grain size and sediment stability are main factors controlling subtidal sand communities (Basson et al., 1977). In the present study, the effects of salinity and sediments

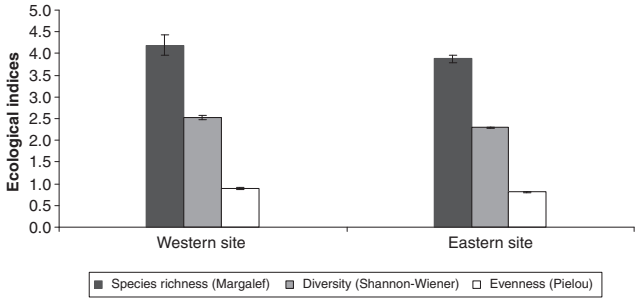

Figure 4 Ecological indices of faunal assemblages in the western and eastern sites.

in governing macrobenthos community patterns in the sampling areas were evidently reflected in the statistical analyses.

Species richness in the Arabian Gulf has been shown to be low in areas where high salinity dominates (Sheppard et al., 1992; Price et al, 2002). High numbers of individual organisms were associated with the eastern site, where the seagrass cover was relatively lower than the western site indicating that additional environmental factors such as salinity are affecting the abundance of macrobenthic assemblages off the western coast of Bahrain.

Conversely, the higher salinity on the western site did not restrict significantly the diversity of macrobenthic assemblages, which could be attributed to the higher densities of seagrasses and finer sediments. This is consistent with the general trend of increasing faunal diversity with decreasing sediment particle in the Arabian Gulf (Coles and McCain, 1990). Higher seagrass cover was reflected in the evenness of the community structure of the western site. Crustaceans only accounted for 5% of the total community population in the eastern site compared with 39% in the western site. Despite the extreme natural environmental conditions including high levels of salinity that may restrict richness of macrobenthos in the Arabian Gulf (Sheppard et al., 1992), seagrass beds maintain high levels of biodiversity and evenness. Therefore, conserving and managing these beds are deemed to be priorities due to the increased human-induced pressures on the coastal and marine environments.

Several implications could be derived from this study to conserve seagrass beds in Bahrain. Human-induced alterations of salinity and sediment characteristics due to hypersaline discharges and the massive reclamation and dredging activities in the marine environment of Bahrain could affect directly or indirectly seagrass beds and their associated biota. Recently, several major economic, housing and recreational projects based on or related to the coastal and marine environment have been undertaken at a rapid rate in Bahrain. Such mega-projects could interfere with water circulation and subsequently alter the salinity (Al-Jamali et al., 2005), which in turn may affect seagrass ecosystems.

Salinity has been shown to increase during the construction phase of major causeways in the Arabian Gulf (Price et al., 1985). Such implications should be incorporated into environmental studies for the proposed new causeway (ca. 40 km) linking between Bahrain and Qatar. This new causeway could alter the current velocities and salinity around Hawar islands. These islands host the most extensive seagrass beds in Bahrain (Phillips, 2003).

Fluctuations in salinity due to brine discharge from desalination plants may affect the marine biota (Miri and Chouikhi, 2005; Abdul-Wahab, 2007). For instance, Gacia et al. (2007) reported that the seagrass *Posidonia oceanica* showed some evidence of salinity stress such as lower rate of leaf growth and increased leaf necrosis as a result of brine discharge off the Spanish coast. In Bahrain, most of the major power and desalination plants are located on the east coast, which may result in cumulative impacts from the brine discharges.

Turbidity and sedimentation are commonly associated with dredging and reclamation activities, which may result in disappearance of seagrass beds from coastal areas mainly due to physical removal and/or burial (Erftemeijer and Lewis, 2006). Although sensitivity to burial varies among species, *H. uninervis* and *H. ovalis* showed 50% shoot mortality at a burial level of 2 cm (Cabaco et al., 2008). Within the last three decades, reclamation activities in Bahrain resulted in adding around 55.5 km^2 to the total land area (CIO, 2003), which could have impacted the surrounding seagrass beds. For instance, Zainal et al. (1993) reported a loss of 10.2 km^2 of seagrass beds on the east coast of Bahrain that were detected from remote sensing imagery

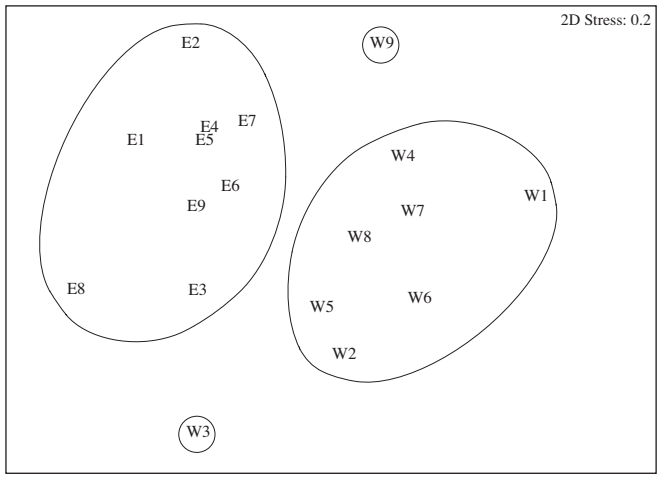

Figure 5 MDS plot for square-root transformed faunal abundance using Bray–Curtis similarity coefficient. Resemblance matrices were clustered to generate similarity of 35% on the MDS. ANOSIM: $R = 0.541$, $P = 0.001$.

between the period of 1985 and 1992, which was mainly attributed to dredging and reclamation activities.

5. Conclusions

Bahraini coastal and marine environments will continue to be a major focus for developmental projects. Therefore, it is important to maintain a balance between such legitimate developments and conserving coastal and marine environments. Seagrass ecosystems should be given a priority in the conservation and management efforts in Bahrain. Introducing effective mitigation measures that associated with environmental impact studies of major coastal and marine projects and enforcing existing regulations related to dredging and reclamation may contribute into conserving these ecosystems.

Additionally, conducting ecological baseline studies, and monitoring programs are essential parts of any effort to conserve seagrass ecosystems in Bahrain. These studies should not be limited to seagrasses, but also extended to investigate other primary producers such as algae and secondary consumers including macrobenthic and fish assemblages, and their interactions (Walker et al., 2001), which could increase the national and international conservational benefits of these habitats.

Acknowledgments

The first author was financially supported by the Master programme in Environment and Sustainable Development at the Deanship of Scientific Research, University of Bahrain. We would like to thank all the staff at the Department of Biology, College of Science, University of Bahrain for their logistical and technical help. Thanks are extended to Mr. Ahmed Al-Shikh and Mr. Ali Mansoor for their help in the field, and Mr. Wisam Mohammed for his help in producing the map using GIS. We are grateful to Dr. David Jones for his constructive comments on the manuscript.

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التغيرات المستحدثة بسموم الأفلاتوكسين ب1 للكروموسومات والجينات الريبوسومية في نبات القمح

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الملخص:

يعد الأفلاتوكسين ب1 أحد النواتج الأيضية الثانوية التي تفرزها بعض الفطريات وهو بدوره يسبب العديد من الأمراض الخطيرة للنبات والحيوان والإنسان. ولقد أكدت الدراسات السيتولوجية والجزيئية التي أجريت في هذا البحث على نبات القمح السمية الجينية للأفلاتوكسين ب 1 تحت التركيزات المختبرة وهي (5 ، 10 ، 15 ، 20 ، 25 ميكروجرام/مل). كما أشارت الدراسات السيتولوجية إلى حدوث أنواع مختلفة من التشوهات في كروموسومات القمح أثناء الانقسام الميوزى؛ فقد تم رصد خلايا تحتوي على كروموسومات متلزنة وأزواج كروموسومات شاردة وجسور كروموسومية وكروموسومات متلكئة وانقسام نووى غير متماثل، وأيضا وجود نويات صغيرة في خلايا الطور البينى. وقد لوحظ أن النسبة المئوية الكلية للتشوهات الكروموسومية تزداد بالزيادة التدريجية في تركيز الأفلاتوكسين ب 1 حتى تصل لأعلى قيمة (2.9%) عند تركيز 15 ميكروجرام/مل ثم تنخفض مرة أخرى إلى (1.7%) عند أعلى تركيز للأفلاتوكسين ب 1 وهو 25 ميكروجرام/مل. وقد أظهرت الدراسات الجزيئية بعض التغيرات على أنماط الفصل الكهربى لحزم الدنا DNA الناتجة من تفاعل البلمرة المتسلسل (PCR) باستخدام الباد ئ الخاص بتتابع دنا الريبوسومى (5S rDNA)، وبينما كان حجم هذه الحزم 100 ،400 ،500 ،800 ،900 زوج من القواعد النيتروجينية في نباتات الكنترول، فقد اختفت بعض هذه الحزم في النباتات المعالجة بالأفلاتوكسين ب1، وهذا يشير إلى احتمال حدوث تغيرات يتم توارثها في تتابع الدنا المسئول عن تكوين الحمض النووى الريبوزى الريبوسومى rRNA.

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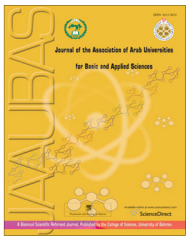

ORIGINAL ARTICLE

Aflatoxin B_1 induces chromosomal aberrations and 5S rDNA alterations in durum wheat

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Available online 9 September 2011

KEYWORDS

Genotoxicity;
Aflatoxin B1;
5S rDNA;
Meiotic chromosomes;
Wheat

Abstract Aflatoxin B_1 is a secondary metabolite of some fungi that causes very serious diseases in plants, animals and humans. Both cytological studies and molecular techniques revealed that AFB_1 at tested concentrations of 5, 10, 15, 20 and 25 μg/ml exhibits genotoxic effect in wheat plants. Several types of chromosomal aberrations have been detected during meiosis; these aberrations include chromosome stickiness, outside bivalents, bridges, laggards, unequal division and micronuclei. The percentage of total abnormalities increased gradually with the increase of toxin concentration but declined again at the highest concentrations. The highest value (2.9%) of abnormalities, detected during meiosis, was in plants treated with 15 μg/ml AFB_1 even so, this value decreased to 1.7% in plants treated with the highest concentration (25 μg/ml) of AFB_1. The 5S primer generated amplified DNA fragments of 100, 400, 500, 800 and 900 bp in control plants. However, some of these fragments were missing or faint in a number of AFB_1-treated plants. Such considerable alterations in DNA profiles of 5S primed amplicons might indicate the generation of various alterations in the inherent property of the 5S rRNA gene sequence.

1. Introduction

Mycotoxins are secondary metabolic products from moulds which can grow on the plant either in the field or during storage (El-Naghy et al., 1991) and are potentially toxic for human beings, animals and plants (Fadl-Allah, 1987 and Helmey, 2003). These toxins are found as natural contaminants in many foodstuffs of plant origin, particularly cereals but also fruits, hazelnuts, almonds, seeds, fodder and foods consisting of or manufactured from these products and intended for human or animal consumption. Mycotoxins can be simply classified according to their major toxic effects. Amongst the groups of mycotoxins, which have been considered being important from a processed-foods and health perspective are the aflatoxins (French Food Safety Agency, Summary report, 2006). Aflatoxins are difuranocoumarin derivatives produced by a polyketide pathway by many strains of *Aspergillus flavus* and *A. parasiticus*; in particular, *A. flavus* which is a common contaminant in agriculture. Aflatoxin B_1 is the most potent natural carcinogen known (Squire, 1981) and is usually the major aflatoxin produced by toxigenic strains. The mutagenicity of aflatoxin B1 to animals and humans is believed to

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Peer review under responsibility of University of Bahrain.
doi:10.1016/j.jaubas.2011.06.002

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involve oxidative activation by a cytochrome P_{450}, in the liver and the kidney. An epoxide is created at the 8, 9 position in AFB_1 that reacts with the N^7-guanine residues of DNA, causing mutations in the tumor repressor gene p53 (Minto and Townsend, 1997).

The structure, organization, and evolution of the 5S rRNA (ribosomal ribose nucleic acid) multi-gene family have been studied in detail in the Triticeae (Dvorak et al., 1989 and Scoles et al., 1988). The 5S rRNA multigenes exist in two sizes: the long unit (ca. 500 bp), consisting of a 120-bp coding and 380-bp spacer region, and a short unit (ca. 400 bp), consisting of a coding region of the same length (120-bp) and a smaller spacer nine region (ca. 280 bp). Plant 5SrDNA is organized in tandemly repeated arrays that occur at one or more chromosome loci (Goldsbrough et al., 1981; and Sastri et al., 1992). The 5S rDNA repeat usually consists of a 120-base pair genic region and a nontranscribed spacer of variable length. The 120-bp genic region is conservative and can be aligned well at broad taxonomic levels (Szymanski et al., 1998). The intergenic spacer region is much more variable among plant taxa and ranges in size from 100 to 700 bp (Cox et al., 1992; and Sastri et al., 1992). The number of repeats per genome can vary from less than 1000 to over 100000 (Schneeberger et al., 1989; Sastri et al., 1992; and Cronn et al., 1996).

The objective of the present study was to determine the involvement of other possible mechanisms in Aflatoxin B_1 (AFB_1) induces toxicity. The applied material was the durum wheat plant (*Triticum durum*) which considered as one of the most important crop plants in Egypt.

2. Materials and methods

Grains of Durum wheat variety Beni-sueif 1) that has been used in this study of this wheat were kindly obtained from Seds Research Center Beni-sueif governorate, Egypt. The pure *Aspergillus* toxin Aflatoxin B_1 (Fig. 1) used in the present study, was obtained from Sigma chemical company, USA. Five different concentrations (5, 10, 15, 20, 25 μg/ml) of this toxin were employed. Aflatoxin B_1 was dissolved in 70% ethyl alcohol and the appropriate dilutions were prepared with sterile distilled water.

2.1. Grain germination

Similar mature grains of durum wheat were selected and surface sterilized using 70% ethanol followed by H_2O_2 for 3 min., rinsed in sterile distilled H_2O three times and allowed to dry on sterile filter paper. Grains were then soaked in sterile H_2O for 24 h and then soaked in the appropriate concentrations of the toxin. As a control, grains were soaked in sterile ddH_2O free of toxin, for the same time. Grains were grown in the field in December 2005 season. Each line was planted as 30 plants/row. Numbers of tillers of 10 randomly chosen plants were scored at flowering stage.

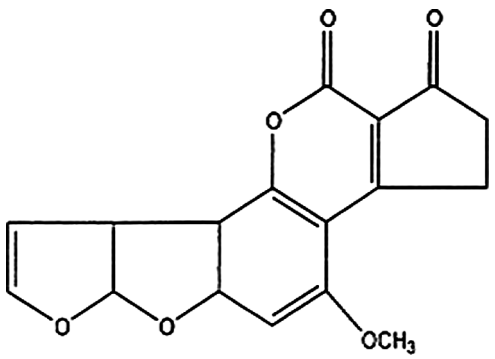

Figure 1 Formula of aflatoxin B_1 (AFB_1).

2.2. Cytological methods

Cytological effects of Aflatoxin B_1 were studied by performing meiotic chromosome analysis at first and second stages of meiotic division. Young flowering buds of wheat were collected at 8–9 o'clock in the morning and immediately fixed in a freshly prepared mixture of ethyl alcohol and glacial acetic acid (3:1 v/v) for 24 h. Pollen mother cells (PMCs) were prepared excised from the fixed anthers and stained with aceto-carmine (1%). Chromosomal aberrations in treated plants compared to the control were scored in at least 3–5 slides of each concentration.

2.3. Molecular studies

2.3.1. DNA extraction

Fresh young leaves of the previously selected 10 plants of each concentration were collected. About 100–300 mg of leaves of each sample were grinded in liquid nitrogen to a fine powder, transferred to 1.5 ml Ependorf tube then 500 μl of Cornel extraction buffer (500 mM NaCl; 100 mM Tris–HCl, pH 8.0; 50 mM EDTA and 0.84% SDS) pre-warmed to 65° C was added. The tubes were placed at 65 °C in water bath for 45 min. After cooling slightly to the room temperature, the specimens were washed by using the same volume (~500 μl) of phenol; phenol: chloroform: isoamyl alcohol (25:24:1) and chloroform: isoamyl alcohol (24:1), respectively at room temperature. About 400 μl supernatant were drawn off into a new 1.5 ml tube containing 1 ml of 100% cold ethanol. DNA was precipitated by centrifugation at 13000 rpm for 10 min then washed with 1 ml 70% cold ethanol. After drying, pellet of DNA was dissolved in 50 μl of re-hydration buffer, and then stored at 4 °C DNA was verified by 1% agarose gel electrophoresis and its concentration and purity were determined with a spectrophotometer at 260 and 280 nm absorbance as mentioned by Sambrook et al. (1989).

2.3.2. PCR conditions (5S primer)

Two specific 5S oligo-nucleotide primers (forward and reverse) were used for screening the 5S rDNA region(s) in the mitotic chromosomes of Durum wheat. The primers were 20-base long with the following sequence:

5′- CGGTGCATTAATGCTGGTAT-3′ forward

5′- CCATCAGAACTCCGCAGTTA- 3′ reverse

DNA amplifications were performed in a final volume of 50 μl containing 25 μl 2× master mix (0.05units/μl Taq DNA polymerase in 2× PCR buffer [4 mM $MgCl_2$ and 4dNTPs (0.4 mM of each)], 10 μM of primer and (1 ng) of DNA template. The final reaction volume was completed to 50 μl using sterilized double distilled water.

The amplifications were carried out in a thermal cycler (Thermo Hybaid, Franklin, USA) programmed for initial preheating period in one step of 5 min. at 94 °C; subsequent 30 cycles of three steps in each, the first step was DNA denaturation

Table 1 The mean number of tillers/plant treated with different concentrations of AFB_1.

Concentrations	Mean number of tillers
Control	5.3
5 μg/ml AFB_1	2.3
10 μg/ml AFB_1	4.6
15 μg/ml AFB_1	4.9
20 μg/ml AFB_1	4.4
25 μg/ml AFB_1	3.7

at 94 °C for 45 s, followed by the step of primer annealing at 56 °C for 45 s and the third step of primer extension at 72 °C for 45 s; subsequent the final cycle in one step of post extension at 72 °C for 10 min. Amplification products were resolved by gel electrophoresis on 1.5% agarose gels in Tris–acetate EDTA (TAE) buffer for 1 h at 80 V. Subsequently, gels were stained with ethidium bromide (0.1 g ethidium bromide dissolved in 10 ml 1× TAE buffer) for 30 min., visualized on UV light and photo-documentation was performed. DNA fragments sizes were estimated by comparison with the standard marker of 1 Kb ladder.

3. Results and discussion

3.1. Germination

The Number of tillers in 10 randomly chosen plants was scored after three months of planting in the field (Table 1). The highest number of tillers was found in control plants whereas, plants treated with 5 μg/ml AFB_1 showed the lowest numbers of tillers. However, the numbers of tillers were increased gradually by increasing toxin concentration up to 15 μg/ml conversely; tillers number was declined again at the highest concentrations. The highest value (4.9) was in plants treated with 15 μg/ml AFB_1 even so, this value decreased to 3.7 in plants treated with the highest concentration (25 μg/ml) of AFB_1. Similar effects of Aflatoxin were reported by Crisan (1973) in *Lepidium sativum*. He found that, concentrations more than 10 μg/ml of aflatoxin, induced the maximal reduction in the rate of growth of hypocotyls after germination. His results were discussed in relation to the effects of Aflatoxin on DNA dependent RNA biosynthesis.

Reduction in number of tillers in plants treated with high concentrations of Aflatoxin may be due to the accumulation of DNA damage in cells, which leads to apoptosis. This may be similar to the effect of Zeralenone, which arrest cell cycle and induces apoptosis in cultured DOK cells as reported by Salwa et al. (2003). She reported that, the apoptotic pathway is the only option for a cell when DNA repair systems are overburdened due to too many damages. Vogelstein and Kinzler, 1992 have shown that among its diverse functions, the p53 gene normally prevents DNA replication in cells that have DNA damage by maintaining the cell in G2/M phase allowing more opportunity for DNA repair. Cells with inactivated p53 might therefore survive abnormally and allow further DNA damage to accumulate (Lane, 1992), a situation, which favours carcinogenesis (Symonds et al., 1994).

Table 2 The frequency and types of chromosomal abnormalities induced during first meiosis in the PMSc of plants treated with different concentrations of Aflatoxin B_1.

AFB_1 Concentration (μg /ml)	Meiosis I															
	Diakinasis			Metaphase I			Anaphase I				Telophase I			Total		
	Normal	Sticky	Out-side	Normal	Sticky	Out-side	Normal	Lag	Bridge	Fragment	Normal	Lag	Fragment	Normal	Abnormal	Aberrations%
Control	252	1	1	150	4	1	52	2	3	1	145	-	-	599	13	2.12
5	114	3	4	310	2	-	68	-	13	-	333	-	-	825	22	2.6
10	142	-	-	188	9	13	103	-	9	-	443	2	-	876	33	3.63
15	240	5	7	145	1	3	102		11		168	3	-	655	30	4.38
20	131	7	2	369	15	20	127	1	17	-	498	-	4	1125	66	5.54
25	165	7	1	235	9	25	33	-	6	-	125	-	-	558	48	7.92

Table 3 The frequency and types of chromosomal abnormalities induced during the second meiosis in the PMCs of plants treated with different concentrations of Aflatoxin B_1.

AFB_1 Concentration (μg/ml)	Meiosis II												
	Metaphase II			Anaphase II				Telophase II		Tetrad		Total		
	Normal	Sticky	Out-side	Normal	Lag	Bridge	Unequal-division	Normal	Micro-nuclei	Normal	Micro-nuclei	Normal	Abnormal	Aberration %
Control	185	1	2	151	–	4	4	207	4	–	–	543	15	2.69
5	36	2	–	38	–	1	–	122	1	80	2	276	6	2.13
10	104	18	–	102	1	4	6	137	1	210	–	553	30	5.14
15	95	21	–	8	1	1	1	285	1	127	2	515	27	4.98
20	210	5	–	72	1	–	3	108	–	–	–	390	9	2.25
25	85	3	8	57	–	3	6	133	1	182	4	457	25	5.18

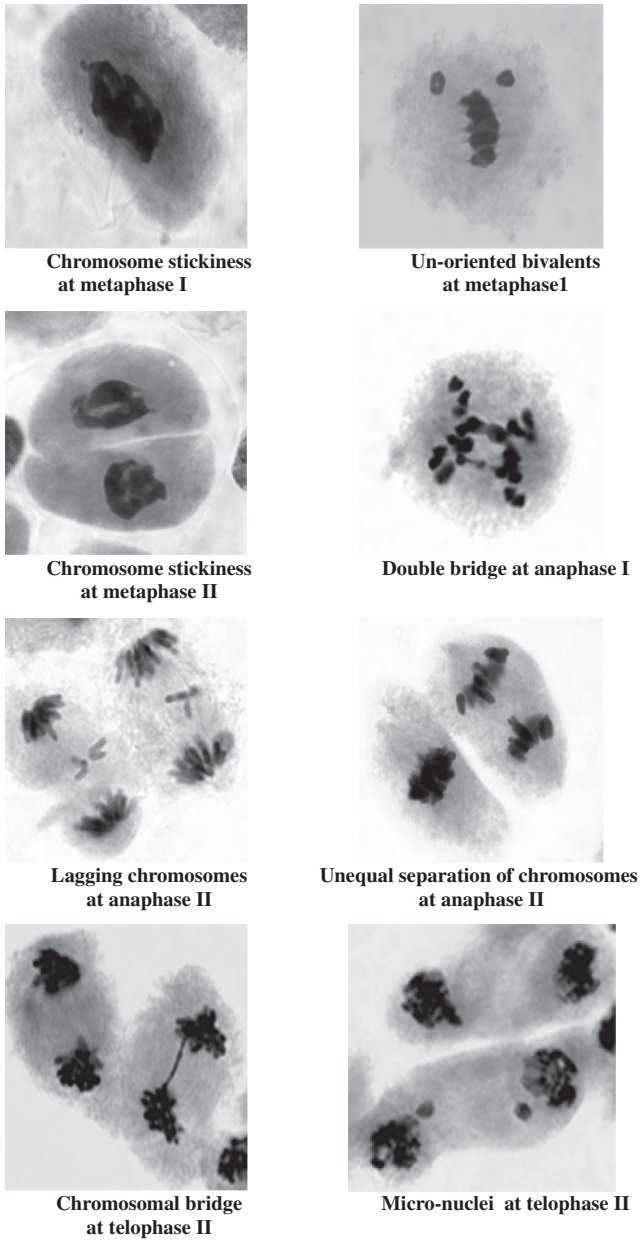

Figure 2 Types of chromosomal aberrations induced by AFB_1 in the PMCs of wheat plants during 1st meiotic and 2nd divisions.

3.2. Cytogenetic effects

The types and frequency of chromosomal aberrations during first and second meiotic division of wheat plants treated with different concentrations of Aflatoxin B_1 were analyzed. At least three slides of each concentration were examined. The common types of chromosomal irregularities, which have been recorded in the present study were stickiness, laggards, bridges, fragments, unequal division (lag-division) and micro-nuclei. The meiotic values of normal cells and types of chromosomal irregularities induced by AFB_1 during the first and second meiotic divisions are shown in Tables 2 and 3, respectively. The results showed that, Aflatoxin B_1 provoked several chromosomal irregularities in wheat at different stages of meiotic division. (Tables 2 and 3).

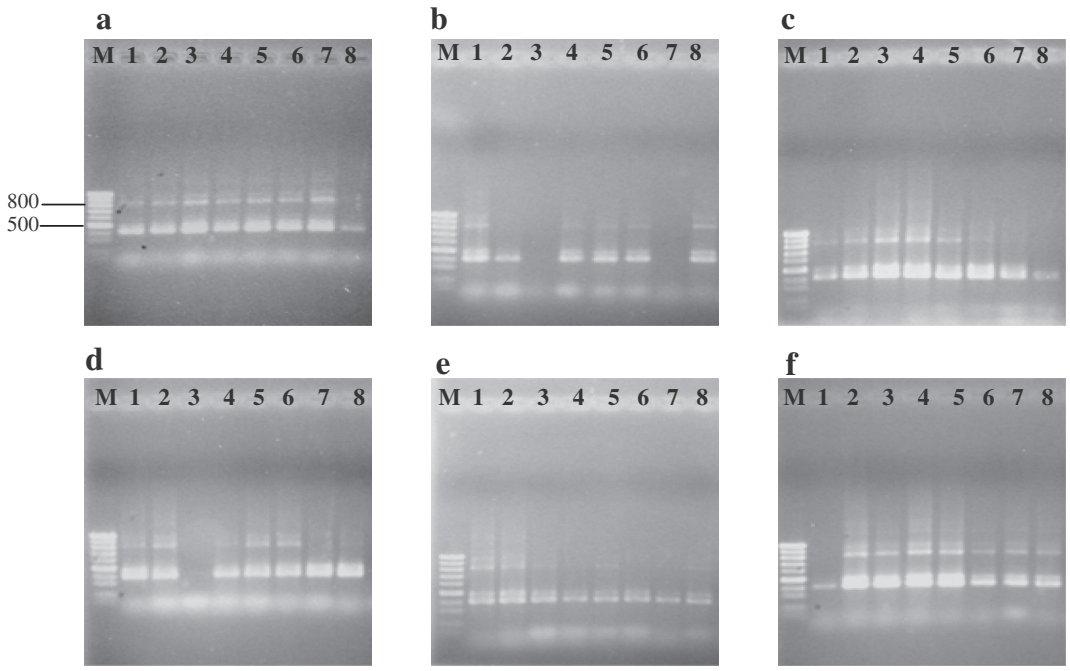

Figure 3 DNA amplification patterns showing the effect of different concentrations of Aflatoxin B_1 on eight wheat plants using the specific 5S DNA primer. (a) Control (b) 5 AFB_1 (c) 10 AFB_1 (d) 15 AFB_1 (e) 20 AFB_1 (f) 25 AFB_1.

Regarding the first meiotic division, data in Table 2 showed that out of 599 PMCs of the control plants, 13 were abnormal with a percentage of 2.12%. Plants treated with 5 μg/ml AFB_1 revealed 825 normal PMCs and 22 abnormal cells with 2.6% abnormalities. The percentage of abnormalities was increased up to 3.63 in plants treated with 10 μg/ml of AFB_1. Plants treated with 15 μg/ml AFB_1showed 4.38% abnormalities. The percentage of abnormalities was further increased up to 5.54% upon treating with 20 μg/ml of $AFB_1$and reached to the highest value of 7.92 in plants treated with 25 μg/ml AFB_1.

In the second meiotic division, control plants exhibit 543 normal PMCs and 15 abnormal cells with a percentage of 2.69% (Table 3). Relating to the control, the lowest concentration (5 μg/ml) of AFB_1did not reveal any effect on the percentage of abnormalities, as it is provoked six abnormal cells out of 276 PMCs in the second meiotic division with a percentage of 2.13%. The percentage of abnormalities was increased to 5.14% in plants treated with 10 μg /ml AFB_1. Plants treated with 15 μg /ml AFB_1showed 4.98% of abnormal cells. Data also revealed that plants treated with 20 μg/ml AFB_1 showed 2.25% abnormal cells. Whereas, the percentage of abnormal cells detected in plants treated with the highest concentration used in this study (25 μg/ml) of AFB_1 was 5.18%. In brief the percentage of chromosomal abnormalities induced by AFB_1 in the first and second meiotic divisions are clearly dose dependant, as it increased as the concentration of the toxin increased and the duration of the treatment prolonged.

Examples of chromosomal abnormalities produced in the PMCs of plants treated with different concentrations of toxin during the 1st and the 2nd meiosis are shown in Fig. 2. Bridges at anaphase could be explained according to the principle that, broken chromosome ends exhibit a tendency to fuse and form dicentric chromosomes (Werner et al., 1992). These dicentrics usually give rise to chromatin bridge-fusion-bridges at anaphase and may break at telophase, thus perpetuating the break (BFB) cycle after fusion of the newly broken chromosome ends.

3.3. Molecular studies

3.3.1. The 5S rDNA specific primer

In the present investigation the specific 5S DNA primer has been used to detect the possible mutagenic effects of the toxins at the genome level in wheat seedlings. The PCR reactions using this primer showed that DNA amplicons posses some variations in the DNA profiles within the examined samples as compared with the control pattern. Such variations include band intensity and appearance of novel bands or disappearance of others. DNA amplification patterns of wheat plants treated with AFB_1 have been shown in Fig. 3.

The number and the intensity of the amplified DNA fragments produced by PCR using 5S primer are listed in Table 4. Plants treated with aflatoxin B_1 showed some changes in the 5S primed amplification profiles as compared with to the control ones. Data in table 4 show that 5S DNA primer generated a band of 100-bp, which is clear in all plants except plant numbered three, seven, eight treated with 5 μg/ml AFB_1, which posses faint bands. However, the amplified fragment of 400-bp was present in all control and treated plant, while it was absent in plant numbered three, seven treated with 5 μg/ml AFB_1 and plant numbered three treated with 15 μg/ml AFB_1.

The 5S DNA primer also displayed a 500-bp band, which was clear in almost all plants while it was faint in the plant No. 2 treated with 5 μg/ml AFB_1 and plant No. 1 treated with 10 μg/ml AFB_1, that was absent in the plant No. 8 of the control, plants numbered three and seven treated with 5 μg/ml AFB_1, plant No. 8 treated with 10 μg/ml AFB_1, plant No. 3 treated with 15 μg/ml AFB_1 and plant No. 1 treated with 25 μg/ml AFB_1. Data also show that, the amplified DNA

Table 4 The presence and absence of amplified DNA fragments generated by 5S DNA primer in wheat plants treated with AFB_1.

Base No.	MW (bp)	Control				5				10														
		1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
5	900	–	–	+	+	–	–	+	–	–	–	–	–	–	–	–	–	–	*	*	*	*	–	–	–
4	800	+	+	+	+	+	+	+	–	+	–	–	–	*	*	–	+	*	+	+	+	+	+	–	–
3	500	+	+	+	+	+	+	+	–	+	*	–	+	+	+	–	+	*	+	+	+	+	+	+	–
2	400	+	+	+	+	+	+	+	+	+	+	–	+	+	+	–	+	+	+	+	+	+	+	+	+
1	100	+	+	+	+	+	+	+	+	+	+	*	+	+	+	*	*	+	+	+	+	+	+	+	+
Base No.	MW (bp)		15							20								25							
		1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
5	900	–	*	–	–	*	*	–	–	+	+	–	–	–	–	–	–	–	+	–	+	+	–	–	–
4	800	*	+	–	*	+	+	–	–	+	+	–	–	+	–	–	+	–	+	+	+	+	+	+	+
3	500	+	+	–	+	+	+	+	+	+	+	+	+	+	+	+	+	–	+	+	+	+	+	+	+
2	400	+	+	–	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
1	100	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

(+) Present and clear (*) present and faint (–) absent.

fragments of 800-bp and 900-bp were amplified in few plants but was missing in many others compared to the control profile.

The presence or absence of a particular DNA band depends on the presence or absence of the DNA sequence to be amplified within the genome (Michelmore et al., 1991 and Abdel-Razik, 1998). Based on such assumption, it could be suggested that treatment of wheat plants with toxins might induce DNA rearrangements or deletions that lead to the observed changes. Yang and Quiros (1993) reported that, the intensity of DNA band depends on the starting copy number of a particular DNA sequence within the genome. Therefore the change in band intensity could be interpreted on the basis of deletion or duplication of some DNA sequences.

Induction of chromosomal aberrations as well as changes in amplified DNA profiles may reflect the direct effect of Aflatoxin B_1 on DNA molecule. In animals, the mutagenicity of aflatoxin B_1 is believed to involve oxidative activation by a cytochrome P450 in the liver and the kidney. Cytochrome P450 enzymes convert aflatoxins to the reactive 8,9-epoxide form (also referred to as aflatoxin-2,3 epoxide in the older literature), which is capable of binding to both DNA and proteins (Eaton and Groopman 1994). Mechanistically, it is known that the reactive aflatoxin epoxide binds to the N^7 position of guanines. Moreover, aflatoxin B_1-DNA adducts can result in GC to TA transversions, (Bennett and Klich 2003). Studies of liver cancer patients in Africa and China have shown that a mutation in the p53 tumor suppressor gene at codon 249 is associated with a G-to-T transversion (Bressace et al., 1991 and Hsu et al., 1991).

DNA adduct formation could be induced in rat tissues following oral administration of acrylamide as reported by Manière et al. (2005). Also, inhalation to low concentrations of 1,3-[2,3-[(14)C]-butadiene provoked DNA-adduct formation in tissues of rats and mice, (Booth et al., 2004). Such effect was reported by Pfohl-Leszkowicz et al. (1995) who studied the genotoxicity of zearalenone, an estrogenic mycotoxin in female mouse tissues.

Acknowledgments

This work was performed in the Molecular Genetics and Cytogenetics Laboratories, Faculty of Agriculture, Minia University. The authors are grateful to Prof. Dr. A. M. Ata for his sincere help.

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تأثير محتوى الكبريتات على الخواص السطحية لحفازات أكسيد القصدير (IV) المدعم بالكبريتات

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الملخص:

في هذه الدراسة تم تحضير حفازات أكسيد القصدير النقية عن طريق إضافة محلول الأمونيا المخفف إلى محلول كلوريد (IV) القصدير حتى الوصول إلى رقم هيدروجيني 8.0 . وبعد غسيل الراسب المتكون عدة مرات والحصول على أكسيد القصدير (جل) وحرقه عند 873 كلفن لمدة 3 ساعات، ثم إضافة حمض الكبريتيك ½ مولاري إلى محلول الجل المتكون قبل عملية الحرق للحصول على حفازات أسيد القصدير المدعم بالكبريتات وبنسب مختلفة من الكبريتات (6، 12، 18 %). بعدها تم توصيف الحفازات بعدة تقنيات منها: التحليل الحراري الوزني والتفاضلي، حيود الأشعة السينية، إمتزاز النتروجين عند 77 كلفن، الأشعة تحت الحمراء باستخدام أقراص بروميد البوتاسيوم بالإضافة إلى قياس الحامضية عن طريق المعايرة الجهدية باستخدام بيوتيل الأمين العادي. وقد أظهرت النتائج أن الجل المتكون قبل الحرق يحتوي على جزيئين من الماء ($SnO2.2H2O$). كما أن إضافة الكبريتات لم تتسبب في حدوث تغيير لطور أكسيد القصدير (التركيب البلوري الرباعي) إلا أنها تسبب انخفاض في حجم الدقائق وبالتالي تحدث زيادة في مساحة السطح. وبزيادة محتوى الكبريتات تزداد كذلك مساحة السطح حتى تصل إلى 88م2/جم لعينة 18%. هذا بالإضافة إلى أن وجود الكبريتات على سطح الأكسيد تظهر تركيبين مختلفين من الكبريتات على السطح كما إتضح ذلك من فحص الأشعة تحت الحمراء. كما أن دراسة الحامضية قد أظهرت أن إضافة الكبريتات للأكسيد تسبب ظهور مراكز حامضية قوية جداً على السطح.

Journal of the Association of Arab Universities for Basic and Applied Sciences (2011) **10**, 15–20

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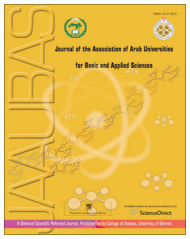

The influence of sulfate contents on the surface properties of sulfate-modified tin(IV) oxide catalysts

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Available online 20 July 2011

KEYWORDS

Tin(IV) oxide;
Sulfation;
Acidity;
Texture

Abstract Tin(IV) oxide catalysts, pure and surface-doped with different loading levels of sulfate, have been prepared and characterized by means of thermal analysis (TGA and DTA), X-ray powder diffraction (XRD), nitrogen adsorption at 77 K, FT-IR spectroscopy using KBr pellets and potentiometric titration using *n*-butylamine. The catalysts were prepared from aqueous solutions of ammonia and tin(IV) chloride, with production of an amorphous precipitate that calcined at 873 K for 3 h to give tin(IV) oxide (SnO_2). The sulfation was carried out by impregnation of sulfuric acidic 0.5 M with tin(IV) hydroxide with different amounts of sulfate (6%, 12% and 18% SO_4^{-2} by weight). Structural investigation of the catalysts by TGA, XRD and N_2-sorption revealed that tin(IV) gel has at least two molecules of water to give the formula $SnO_2{\cdot}2H_2O$ and the addition of sulfate does not modify the crystalline structure of tin(IV) oxide (tetragonal phase) but decreases the crystallite size and, consequently, increase the specific surface area. The increase in loading level of sulfate resulted in increase in specific surface area of the catalysts. Acidity measurement by potentiometric titration using *n*-butylamine show that the addition of sulfate can increase the acidity of tin(IV) oxide and all sulfated tin(IV) oxide having strong acid sites. Moreover, FT-IR spectra expose that sulfated tin(IV) oxide has two different structures of sulfates.

1. Introduction

Tin(IV) oxide is an active catalyst for many reactions due to acidic, basic, oxidizing and reducing surface properties. It has become well established that the performance of a heterogeneous catalyst depends not only on the intrinsic catalytic activity of its components, but also on its texture and stability. One of the most important factors in controlling the surface properties of a catalyst involves the correct choice of the additives. The adsorption of various anions (Mekhemer, 2005), particularly sulfate or phosphate anions, onto oxide has been attempted as a means of improving their catalytic activity. The increase in activity is believed to arise from increase in the surface acidity of the modified oxide (Clearfield et al., 1994). Modification of metal oxides with sulfate anion can generate a strong acidity, even stronger than 100% sulfuric acid and hence they become superacid catalysts that are useful in reactions like isomerizations, low temperature esterification, alkylation and cracking (Jyothi1 et al., 2000; Song et al., 1996). Sulfated tin(IV) oxide is one of the candidates for

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Peer review under responsibility of University of Bahrain.
doi:10.1016/j.jaubas.2011.06.003

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having the strongest acidity on the surface. The acid strength is reported to be higher than that of sulfated zirconia (El-Sharkawy et al., 2007; Hino et al., 2007). Ceramics acid of tungstated tin(IV) oxide prepared by Arata is more active than aluminosilicates for the esterification of *n*-octanoic acid with methanol (Hino et al., 2006). Other application of CO oxidation over SnO_2 and Pd/SnO_2 catalysts was reported by (Sasikala et al., 2000), whereas organic syntheses catalyzed by superacidic metal oxides: sulfated zirconia and related compounds were reviewed (Arata, 2009).

Nevertheless, papers concerning sulfated tin(IV) oxide catalyst have been quite few because of difficulty in preparation, compared with the easy preparation of other sulfated oxides. This study aimed to prepare tin(IV) oxide catalysts acidified with sulfate anions and study the effect of sulfate amounts on the surface properties of it. Tin(IV) oxide was prepared from stannic chloride and has been sulfated using sulfuric acid with different loading levels of sulfate (6%, 12% and 18% SO_4^{-2}). These sulfated catalysts were characterized using different tools to know the effect of sulfate levels on the surface properties of the SnO_2 catalysts.

2. Experimental

2.1. Materials

Tin(IV) oxide gel, used as a precursor of sulfated and promoted oxide, was prepared by the method described in the literatures (Matsuhashi et al.2001a,b). In brief, by a slow dropwise addition of a 1:1 aqueous ammonia to a 0.3 M aqueous solution of tin(IV) chloride ($SnCl_4{\cdot}5H_2O$) AR grade, BDH product (England), with a continuous stirring till pH = 8 is reached. The white precipitate was left over-night before being filtered and washed thoroughly with 2% CH_3COONH_4 solution until all chloride was eliminated (silver nitrate test), and then dried at 383 K till constant weight is obtained. The dried material was ground to 250 mesh size and kept dry over P_2O_5 desiccator. The dry gel thus obtained is denoted in the text as TH and it was used as a precursor for preparation of SnO_2 and modified SnO_2 catalysts. Pure tin oxide, SnO_2, was obtained from the dried gel (TH) by calcination at 873 K for 3 h. The resultant oxide, SnO_2 (XRD verified vide infra), was designated in the text as TO. Sulfated SnO_2 samples were prepared by impregnation of $SnO_2{\cdot}xH_2O$ (TH) gel with the appropriate amount of 0.5 M H_2SO_4 solution to obtain 6%, 12% and 18% SO_4^{-2} by weight. The resultant was dried at 383 K for 24 h, followed by calcination at 873 K for 3 h and the products were designated as *x*STO (where x = 6%, 12% and 18%).

2.2. Apparatus and techniques

2.2.1. Thermal analysis

Both TGA and DTA were performed between room temperature and 1273 K in a static atmosphere of air, using V2-2A DUPONT 9900 thermal analyzer. The rate of heating was standardized at 10 K min^{-1}, and small portions (5–15 mg) of the sample were used in TG measurements.

2.2.2. X-ray powder diffractometry

XRD diffractograms were recorded for all samples using a model JSX-60PA JEOL diffractometer (Tokyo, Japan) and CuKα radiation (λ = 1.5418 Å). The generator was operated at 35 kV and 20 mA. The samples were scanned in the range of 2θ = 10–70° at a scanning speed of 6 min^{-1}. For identification purposes, diffraction patterns (I/I°) versus d spacing (Å) were matched with the relevant ASTM standards (Frank, 1981). The crystallite size D of the samples were calculated using the Scherrer's relationship (Klug and Alexander, 1970):

$$D = \frac{k\lambda}{\beta \cos\theta} \quad (1)$$

where K is the crystallite shape constant ($\approx$1), λ the radiation wavelength, β the line breadth (radians) and θ is the Bragg angle.

2.2.3. Nitrogen sorption measurement

Full nitrogen adsorption/desorption isotherms at 77 K were obtained using a NOVA 2200 (version 6.10) high-speed gas sorption analyzer (Quantachrome Corp., Boynton Beach, FL, USA). The calcined samples were first outgassed at 470 K for 1 h. Twenty-four-point adsorption and desorption isotherms were obtained, from which BET surface areas were derived using standard and well-established methods (Sing et al., 1985; Webb and Orr, 1997).

2.2.4. FTIR measurement

A very small amount of finely ground solid sample (5–10 mg) is intimately mixed with powdered KBr (90 mg) and then pressed in a 7 mm die under high pressure. IR analyses of the catalysts were carried out over the frequency range of 4000–500 cm^{-1} using a Nicolet 380 FT-IR spectrophotometer with 4 cm^{-1} resolution.

2.2.5. Acidity measurement

The total acidity of the solid samples under investigation was measured by means of potentiometric titration (El-Sharkawy et al., 2007; Rao et al., 2006). The solid catalyst (0.1 g) was suspended in 10 ml acetonitrile (Merck), and agitated for 4 h. Then, the suspension was titrated with 0.1 N *n*-butylamine in acetonitrile at 0.10 ml min^{-1}. The electrode potential (Ei) variation was measured with SevenMulti, METTLER-TOLEDO, GMBH, Switerland. Cid and Pecci (1985) made a scale of acid strength measurement as follow: Ei > 100 mV for very strong sites; 0 < Ei < 100 mV for strong acid sites; −100 < Ei < 0 mV for weak sites; and finally Ei < −100 mV for very weak sites.

3. Results and discussion

3.1. Thermal analysis

From the TG profile of the precursor tin gel ($SnO_2{\cdot}xH_2O$), Fig. 1a, it can be seen that there are two mass loss steps in the temperature range RT-1273 K. The first step ends at 423 K with loss of mass ~3.6% and accompanied by an endothermic peak at 380 K, is attributed to the loss of volatile materials like physisorbed water (Magnacca et al., 2003). The second one begins just after the first (680 K) bringing the mass loss to ~19.9%, is assigned to dehydroxylation processes (DTA exothermic peak at 643 K) (Mekhemer et al., 2005; Wang and Xie, 2001). Theoretical mass loss of tin gel as stoichiometrically approaching $SnO_2{\cdot}2H_2O$ to anhydrous SnO_2 is 19.3% (Scheme 1). The experimental mass loss is

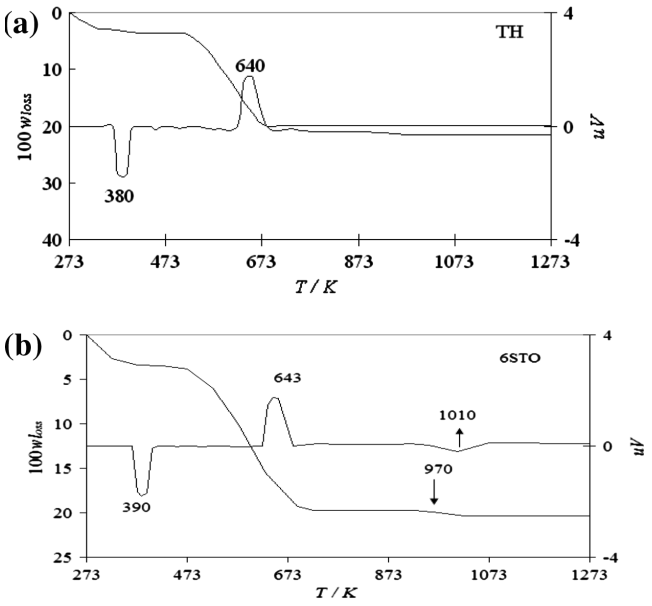


Figure 1 TGA and DTA profiles for (a) tin(IV) gel (TH), and (b) sulfated tin(IV) oxide (6STO), w_{loss} is the mass fraction.

$$SnO_2.2H_2O \xrightarrow{\triangle} SnO_2 + 2H_2O$$

Scheme 1

~19.9% hence this implies that the tin gel sample approaches the suggested formula. Therefore, the exhibited thermal events involve in general elimination of water of different strengths and a final formation of SnO_2, which agree with the XRD data. As to sulfated sample, 6STO, there are three mass loss steps, Fig. 1(b). The first two steps, ends at 473 K (endothermic peak at 393 K) and 643 K (exothermic peak at 693 K), are attributed to removal of adsorbed water and dehydroxylation process of tin gel. The mass loss at higher temperature began at 973 K and ended at 1053 K and accompanied by an endothermic peak at 1013 K, is attributed to decomposition of sulfate groups (Khalaf, 2009; Reddy et al., 2006).

3.2. X-ray diffraction

The XRD diffractograms of the sulfated samples with different loading levels of sulfate (6STO, 12STO and 18STO) calcined at 873 K are shown in Fig. 2. The main conclusion is that all patterns are characteristic of pure SnO_2 phase (ASTM card No. 41–1445) with tetragonal rutile structure at $2\theta = 26.54$, 33.82 and 51.74. The intensities of the bands characteristic for SnO_2 (TO) gradually decreased, while the width of the reflections is considerably broadened, indicating a small crystalline domain size (Khder et al., 2008). To know the role of sulfate content on the crystallinity of the samples quantitatively, their mean crystallite sizes were calculated from the broadening of the strongest peak of the samples and based on Scherrer equation (Table 1). The addition of sulfate was associated with a decrease in crystallite size to become 88, 50 and 31 Å for 6STO, 12STO and 18STO, respectively. This may be attributed to the sulfate groups that remain bounded at the surface of the

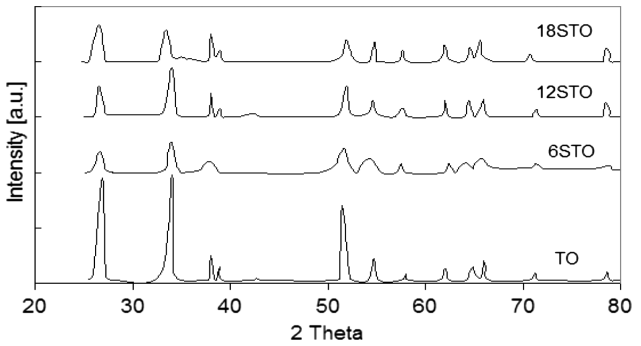


Figure 2 X-ray powder diffractograms for *x*STO in comparison with TO catalysts.

samples and inhibit the growth of SnO_2 crystallites, agreeing thus with the other transition metal oxides, i.e. TiO_2, ZrO_2 and Fe_2O_3 (Khder et al., 2008). The decrease in the crystallite size can be explained by the hypothesis that the bulky sulfate groups on the surface of SnO_2 particles prevent their agglomeration during calcination (Jogalekar et al., 1998).

3.3. Surface texture

The nitrogen sorption isotherms at 77 K of pure oxide (TO) and sulfated samples (6STO, 12STO and 18STO) were shown in Fig. 3. From this figure, it is clear that all isotherms are belonging to Type IV according to BET classification (Brunauer et al., 1938) and displays hysteresis loop of Type H3 mixed with some of H2 in case of 18STO sample according to the IUPAC classification (Sing et al., 1985). All the samples have a close closure point at $P/P_o = 0.4$. This may infer that the monolayer is completed slowly with contribution of micropores. The more pronounced knee on the adsorption isotherm of 6STO sample is reflected on its higher C_{BET} value (Table 1). This value greatly affects the shape of the isotherm in the low-pressure region whatever is the V_m value. Larger hysteresis loops can be noticed for TO, 6STO and 12STO samples.

Values of S_{BET}, S_t, V_P, r_P and C_{BET} for all samples are summarized in Table 1. It is apparent that S_{BET} and porosity are not similar for all samples, most likely due to variation in crystallite size. The good agreement between the S_{BET} and S_t values (Table 1) for all samples, reflects the higher accuracy of the BET-C determination and, consequently, the appropriateness of the reference Va-t curves (Gregg and Sing, 1982). From these data, it is clear that pure TO sample has low specific surface area ($S_{BET} = 11\ m^2\ g^{-1}$) which agrees with previous data (Harrison, 1989). The addition of su1fate into crystalline oxide resulted in a gradual increase in the surface area for the samples 6STO, 12STO and 18STO to become 35, 60 and 88 $m^2\ g^{-1}$, respectively. These data agree with the data obtained from XRD results, which complied in Table 1 also, in which the sulfated samples have small crystallite size (88, 50 and 31 Å for 6STO, 12STO and 18STO, respectively) that affect on the specific surface area (Khalaf, 2009).

The pore size distribution (PSD) curves for the samples, Fig. 4, show that the PSD lies between micropores and mesopores range. Three main peaks at < 20 Å were observed for all samples; in addition, some peaks were shown at radius higher than 20 Å. This means that the porosity of TO is

Table 1 Nitrogen sorption analysis data.

Sample	S_{BET} m^2g^{-1}	C_{BET}	S_t[a] m^2g^{-1}	S_S[b]	S_{cum}[c]	Vp_{cum}[d] cm^3g^{-1}	Vp_{tot}[e] cm^3g^{-1}	Aver r_P[f] Å	Crystallite size[g] (Å)
TO	11	14.7	10.3	10.8	6.9	0.016	0.0325	17.5	185
6STO	35	2.7	34	34.6	17.4	0.036	0.0387	18.5	88
12STO	60	3.5	59	58.8	19	0.035	0.048	13.5	50
18STO	88	4.0	87.7	86.5	41	0.066	0.1036	12.7	31

[a] The standard isotherm used in each case was selected according to (Sing et al., 1985).
[b] α_S Surface area.
[c] Cumulative surface area.
[d] Cumulative pore volume.
[e] Total pore volume at $P/P^o = 0.99$.
[f] Mean pore radius at the peak of the distribution curves.
[g] Obtained from XRD data.

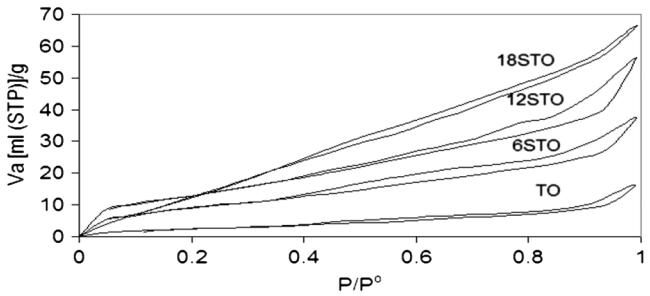


Figure 3 Nitrogen sorption isotherms for *x*STO in comparison with TO; V_a is measured at STP.

micropores in addition to little amount of mesopores. However, sulfated samples show a mix between mesopores and micropores in agreement with the findings of the *t*- and α_S-methods. Then, the incorporate of sulfates onto tin(IV) oxide can modify its porosity.

3.4. FTIR spectroscopy

Fig. 5 shows the IR spectra of sulfated tin(IV) oxide (*x*STO) in comparison with pure tin(IV) oxide (TO). From these spectra, the presence of sulfate groups was confirmed by the bands at 1382, 1190, 1155 and 1079 cm^{-1}. The first band (at 1382 cm^{-1}) characteristic of the surface sulfate species having S═O covalent bonds (Mekhemer et al., 2005). The other bands at 1190, 1155 and 1079 cm^{-1} are due to the asymmetric and symmetric stretching frequency of the O═S═O and O–S–O group (Mekhemer et al., 2005; Clearfield et al., 1994). Thus, it is evident that the IR spectra of both types of sulfated samples show very similar spectral features in the SO stretching region. These features are in general attributed to triply bridging sulfate and bridged bidentate sulfate, as represented in structures (I) and

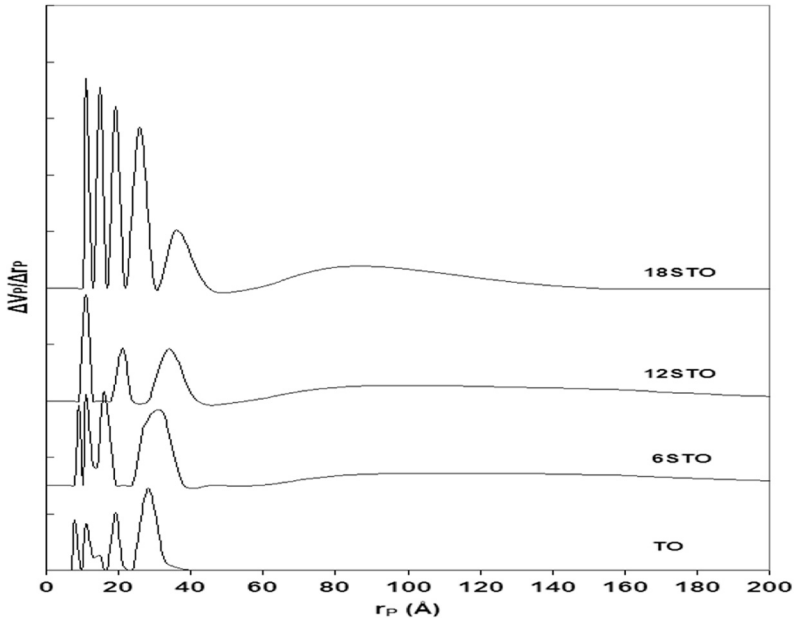


Figure 4 Pore size distribution curves for *x*STO in comparison with TO catalysts; r_p true radius and $\Delta V_P \cdot \Delta r_P^{-1}/cm^3\ g^{-1}\ Å^{-1}$ is the ratio between the volume decrease (ΔV_P) in $cm^3\ g^{-1}$ and the decrease in pore radius (Δr_p^{-1}) in $Å^{-1}$.

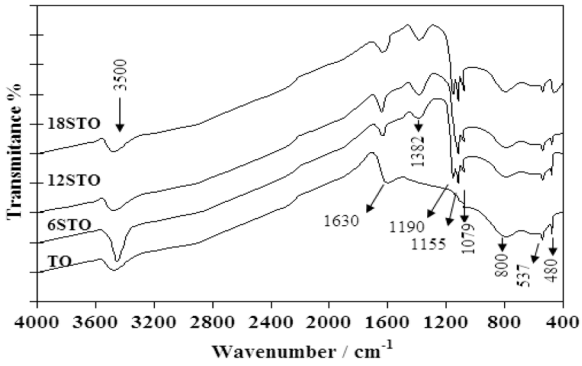


Figure 5 FTIR spectra for *x*STO in comparison with TO catalysts.

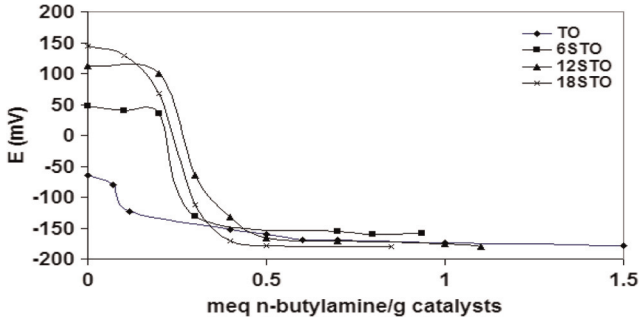
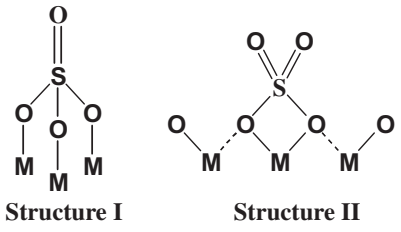


Figure 6 Potentiometric titration curves for *x*STO in comparison with TO catalysts; E is the electrode potential (mV).

(II) where M is Sn ion (Jin et al., 1986; Bensitel et al., 1988). The only effect of sulfate loading resulted in increase in the band intensities of sulfate group.



3.5. Acidity of the catalysts

Total number of acid sites and their relative strength, for the catalysts under investigation, can be measured by means of a potentiometric titration with 0.1 N *n*-butylamine. To explain the obtained results, it was suggested that the initial electrode potentional (Ei) indicates the maximum acid strength of the sites. The value of meq amine/g solid, where the plateau is reached in titration curves (Fig. 6), indicates the total number of acid sites (El-Sharkawy et al., 2007; Cid and Pecci, 1985). Table 2 shows the potentiometric titration results for all samples. From these results one can conclude that the TO sample has weak acid sites and its maximum strength is equal −65 mV. The addition of sulfate to SnO_2 can increase the acidic strength and create strong and very strong acid sites on the surface of SnO_2 to become +47, 112 and 144 mV for the samples 6STO, 12STO and 18STO, respectively.

Table 2 Potentiometric titration data for the catalysts.

Sample	Ei (mV) = maximum acid strength	No of acid sites (meq g^{-1})
TO	−65	0.11
6STO	+47	0.30
12STO	+112	0.40
18STO	+144	0.40

4. Conclusion

The obtained results show that the SnO_2 gel has at least two molecules of water to give the formula $SnO_2{\cdot}2H_2O$ and the addition of sulfate has no effect on the crystalline phase of SnO_2 (tetragonal phase) but decreases the crystallite size and, consequently, increase the specific surface area. The specific surface area is increased by increasing the loading of sulfate. FT-IR spectra expose that sulfated SnO_2 has two different structures of sulfates. Moreover, the incorporation of sulfate onto SnO_2 can increase its acidity and creates strong acidic sites.

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تراكم العناصر الثقيلة في بعض نباتات قطاع غزة–فلسطين والخواص الفسيولوجية لنبات السبانخ

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الملخص:

تمت هذه الدراسة بهدف تقييم بعض العناصر الثقيلة (كادميوم, رصاص, زنك وحديد) في التربة وبعض نباتات شمال قطاع غزة– فلسطين المزروعة عشوائيا في تلك المناطق. ولقد أظهرت نتائج هذه الدراسة أن تركيز هذه العناصر في التربة، وكذلك في النباتات، كان في الحدود المسموح بها عالميا بإستثناء عنصر الرصاص الذي كان مرتفعا وقد فاق الحدود المسموح بها في منطقة المنطار ومركز مدينة غزة. كما استهدف هذا البحث دراسة تأثير تلوث التربة بالكادميوم والزنك على صفات النمو المختلفة والصبغات النباتية لنبات السبانخ المزروع في أصص. ولقد أظهرت نتائج هذه الدراسة أن إضافة الكادميوم أدت إلى تقليل معظم صفات النمو المدروسة وكذلك تقليل تركيز كلوروفيل أ وتركيز العناصر. كما أن إضافة الزنك مع الكادميوم قد أدت إلى تقليل سمية الكادميوم لمعظم صفات النمو المدروسة. كما لوحظ أن للزنك تأثيراً ايجابياً على معظم صفات النمو والصبغات النباتية ، عند استخدامه منفرداً– باستثناء كلوروفيل ب – وعلى تركيز العناصر حيث ارتفع تركيزها باستثناء انخفاض تراكم عنصر الكادميوم.

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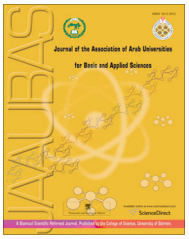

ORIGINAL ARTICLE

Accumulation of heavy metals in crop plants from Gaza Strip, Palestine and study of the physiological parameters of spinach plants

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Available online 22 July 2011

KEYWORDS

Heavy metals accumulation;
Cadmium;
Zinc;
Spinach plant;
Physiological parameters;
Soil

Abstract Measurements of Pb, Zn, Cd and Fe concentrations in the soils and accumulation in edible parts of several crop plants (spinach, wheat, strawberry, carrot, onion, squash, cabbage, potato, faba bean and cucumber) grown in three sites of the northern area of Gaza Strip, Palestine, revealed: (1) Concentrations of metals were in normal range in soil, except for lead concentrations which in some samples were higher, especially in the sites of Al-Monttar and Gaza city center. (2) Accumulation of heavy metals by the crop plants was within normal ranges, except for lead concentration which exceeded normal ranges, yet not reaching toxic levels in all plants but the onion bulb which reached toxic level. (3) Cadmium was concentrated at equal levels in different soil samples, while its accumulation in plant samples was very low and sometimes was not detectable. Measurements of physiological attributes of spinach plants revealed: (1) Growth characters such as root length, shoot height, fresh and dry weights of shoot and root were decreased with increasing Cd soil addition either alone or combined with Zn soil addition at all levels. (2) Plant pigments such as chlorophyll a, chlorophyll b and total carotenoids significantly decreased, with increasing Cd soil addition either alone or combined with Zn at all levels, except for chlorophyll a which increased with increasing Zn soil addition, with some exceptions. (3) Zn addition was highly correlated to growth characters, as well as when combined with Cd at different levels may be overcome the toxicity of Cd on growth characters, mineral concentrations and chlorophyll a content.

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Peer review under responsibility of University of Bahrain.
doi:10.1016/j.jaubas.2011.06.001

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1. Introduction

The Gaza strip–Palestine area is about 360 km^2. It is situated in the south part of Palestine and southeast of the Mediterranean sea. An estimate of 1.6 million people live in the Gaza Strip. This area, being one of the most densely populated areas in the world with limited and deteriorated resources, has already started to suffer the consequences of environment quality deterioration. The situation at the Gaza Strip is below the desired standard, which is attributed to the absence of environmental legislation and public awareness. One of the most

important air pollutants near the city center of Gaza is thousands of motor vehicles commuting every day. Trace metals released in the environment may be considered a hazard to the natural biological system and human health. Plant and soil surfaces are the major sink for airborne metal. Moreover, plants form the basis of food chains by which bio-toxic trace metals are transmitted to man (Alfani et al., 1996).

Improvement of the nutritional quality of our food supply, especially with respect to essential nutrient minerals, could be an important goal of vegetable crops. Cd is classified as probable human carcinogen by inhalation; however, only limited data are available to determine if it causes cancer in humans. The total Zn content in adult human tissue is 2–4 g. The daily requirement of 6–22 mg is provided by a normal diet. Salim et al. (1992) treated the carrot plants by Cd and Pb foliar or root application of 0, 10 and 90 ppm Cd and 0, 18 and 80 Pb. Cd toxicity was more obvious than Pb toxicity and these symptoms were more severe in foliar treated than in root treated plants. Cd application decreased the dry weight of whole plants, shoot and roots, when compared with untreated control plants. William et al. (1977) reported that Cd in soils may be unavailable to plants by the application of hydrated lime and consequently involves the infection of toxicity. Liming is effective because cadmium probably forms an insoluble precipitate with hydroxide and thus becomes unavailable to plants. Somashekaraish et al. (1992) suggested that the inhibition of chlorophyll synthesis by Cd is achieved both by reaction with constituent biosynthetic enzymes as well as peroxide mediated degradation.

Zinc plays essential metabolic roles in the plant, the most significant of which is its activity as a component of a variety of enzymes, such as dehydrogenases, proteinases, peptidases and phosphohydroleases. Other functions related to the metabolism of carbohydrates, proteins, phosphates, RNA and ribosome formation. Baccio et al. (2005) have pointed out that transition metals such as Zinc are essential micronutrients for many physiological processes, but they become toxic at elevated levels, Zinc is one of the most abundant trace heavy metals present in agro-ecosystems. Misra et al. (1994) have mentioned that when *Vicia faba* seeds are treated with 0–10 mg l^{-1} solution of Zn chloride, Zn treatment increases its radical length at low concentrations, but are inhibitory at high concentrations.

The aim of this research was to study the elemental (Pb, Zn, Cd and Fe) concentration on different soils in three sites of the northern area of Gaza Strip, Palestine. The ability of some crop plants (spinach, wheat, strawberry, carrot, onion, squash, cabbage, potato, faba bean and cucumber) grown in the above soils to accumulate different metals was studied. Effects of different levels of Zn and Cd soil addition on the morphological parameters (root length, shoot height, fresh and dry weights of shoot and root, plant pigments such as chlorophyll a, chlorophyll b and total carotenoids) of spinach plants were also examined.

2. Materials and methods

Plant and soil samples were collected in 2006 from the Northern area (120 km^2) of the Gaza Strip, Palestine. The study area has several anthropogenic influences which could be divided into three different sites, (industrial, urban and rural site (Fig. 1), all lying between Israeli borders and the Mediterranean sea. Four samples were collected from each site of the study area, (A, B and C, Fig. 1). Samples of spinach, wheat, strawberry and carrot were collected from Beit-Hanon and Beit-Lahya, industrial and rural site (A). Onion, squash, spinach and cabbage were collected from Al-Monttar and Gaza City Center, industrial and urban site (B). Potato, carrot, faba bean and cucumber were collected from Al-Zytoon and Shakh Ejleen, rural and urban site (C).

All plant samples were taken at the flowering stage, were washed in water, dipped in distilled water and divided into their parts. They were then oven-dried at 70 °C for two days. Soil samples were also oven-dried at 40 °C for two days. Available Fe, Pb, Zn and Cd were extracted by DTPA according to Lindsay and Norvell (1978) and estimated by Atomic Absorption Spectrophotometer GBC 939.

Plastic pots of 80 cm length, 20 cm width and 25 cm depth were used at the pot experiments which carried out in the open field of the Agriculture Research Center (Ministry of Agriculture), Beit-Lahya City, during two successive seasons (2006 and 2007). Each pot was filled with 25 kg soil obtained from the Agriculture Research Center. Spinach seeds (*Spinacea oleracea* var. balady) were used, sown on the 17th of February in the first season (2006) and in the 18th of January in the second season (2007). Each pot received 22 g of ammonium sulfate, 18 g of potassium sulfate and 15 g of calcium superphosphate. The fertilizers were applied to the plants as soil dressing at three doses/season, the first dose was 15 days after seedling emergence and the second and the third dose were applied 15 days in time intervals.

In the second season, before seed sowing, four levels of Cd (0, 10, 20 and 40 mg kg^{-1}) were added to the soil, in contrast to the first season, where three levels of Cd (10, 20 and 40 mg kg^{-1}) were added in the form of cadmium sulphate salt. The pots were divided into four groups in the second season and into three groups in the first season. The first group received the normal level of fertilizers as mentioned above, but

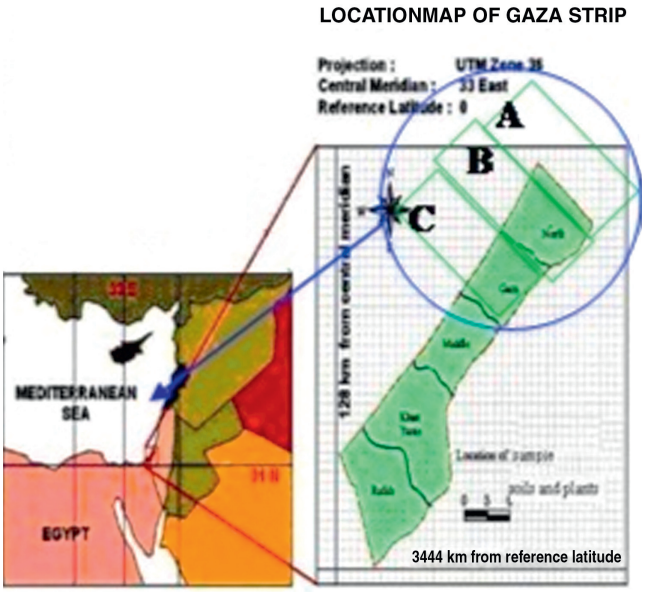

Figure 1 The three sampling sites at the northern area of the Gaza Strip, Palestine.

without any other soil additions. The second, third and fourth group were similar to the first group but the pots received Zn with the soil addition before sowing in the form of Zinc sulphate at the rates (0, 100, 200 and 400 mg kg^{-1}). Therefore, there were three and four levels of Cd soil addition in the first and second season respectively, while four levels of Zn soil addition in the two seasons served as sub-treatment. Sixteen treatments in the second season and 12 treatments in the first season were examined. Two samples were taken from each treatment after 30 and 60 days from sowing and the plant was divided into roots and leaves. The following measurements were recorded: root length (cm), shoot height (cm), fresh and dry weights of the shoot and root (g) and leaf area were estimated in the two seasons, according to the methods described by Hunt (1978). Fe, Zn, and Cd were determined in the two samples (70 and 140 days after sowing) for root and shoot in the 2nd season. Chlorophyll and total carotenoids (mg/g FW) were extracted from the fresh leaves of plants by acetone 80% and were calorimetrically determined according to the method described by Hoyden (1965). Data were statistically analyzed by using factorial experiments and the means of different treatments were compared using the least significant different test (L.S.D.) at 0.05 level of probability in the two samples as average of two successive seasons (Snedecor and Cochran, 1980).

3. Results and discussion

3.1. Bioaccumulation of heavy metals in crop plants

Data in Table 1 reveal that in site A (Beit Hanoun and Beit Lahya site), the extractable mean values of heavy metals (Pb, Zn, Cd and Fe) were 9.9, 41.2, 0.0075 and 333.6 mg kg^{-1}, in site B (Monttar and Gaza City Center site) 20.9, 42.9, 0.0075 and 186.0 mg kg^{-1}, while in site C (Al Zytone and Shakh Ejleen site), 16.3, 53.9, 0.00 and 248.9 mg kg^{-1}, respectively. Similar results were obtained by Shomar (2006) for all metals except for Fe. The author mentioned that in the open farm soils of the Gaza Strip the total Pb, Zn, Cd and Fe concentrations were 32, 50, 0.0052 mg kg^{-1} and 2.2% respectively, while metal concentrations in the strawberry farm soils of the Gaza Strip were 61, 60, 0.0067 mg kg^{-1} and 1% for iron, respectively (1% = 10.000 ppm). The concentration of heavy metals (Cd, Zn, Pb and Fe) are in the normal range in soils, except concentration of lead in some samples which were higher especially in area **B** (Al Montar and Gaza Center) as well as the concentration of iron in area **A** (Beit Hanon and Beit Lahya).

As far as the heavy metals concentration in the plants growing in the study area is concerned, data in Table 1 indicate that Pb concentration in plants is dependent on plant organs. The highest concentration of Pb in site A was 22.1 mg kg^{-1}, recorded in the root of carrots, while the lowest concentration was 12.1 mg kg^{-1}, recorded in strawberry (fruit). Pb concentration in the edible parts of the plants studied can be arranged in the following order: carrot (root), spinach (leaves), wheat (shoot) and strawberry (fruit). Moreover, the highest Pb concentration in site B was 95.0 mg kg^{-1} and was detected in the bulb of the onion, while the lowest concentration (0.11 mg kg^{-1}) was recorded in spinach leaves. Values can be arranged in the following order according to Pb concentration in site B: onion (bulb), cabbage (leaves), squash (fruit) and spinach (leaves). At site C, the highest Pb concentration (12.5 mg kg^{-1}) was recorded in potato tuber, while the lowest Pb concentration (10.1 mg kg^{-1}) was detected in cucumber (fruit). The values of Pb concentration in site C can be arranged in the following order: potato (tuber), faba bean (seeds), carrot (root) and cucumber (fruit). In this respect, one may argue that the highest Pb concentration is recorded in root, bulb and tuber, when compared with another parts of plants (leaves, fruit and seeds). Moreover, it is important to mention that the increase in Pb concentration in the different organs of plants among the three different sites can be arranged in the following order: onion bulb, carrot roots, wheat shoots, potato tuber, strawberry fruit, faba bean seeds, cucumber fruit, spinach leaves, cabbage leaves and squash fruit. In this respect, Bowen (1997) reported that the normal range of Pb in edible vegetables range between 0.20 and 20 mg kg^{-1}. Thus, based on these data, it can be suggested that Pb

Table 1 Heavy metal concentration (mg kg^{-1}) in the soil and plants growing in the study area.

Study areas	Sites	Name and organs of plant	Metals concentration (mg kg^{-1})							
			Plants (dry matter)				Soils			
			Pb	Zn	Cd	Fe	Pb	Zn	Cd	Fe
Beit Hanon and Beit Lahya	A1	Spinach (leaves)	15.5	32.0	0.0000	19.4	18.1	46.8	0.0200	711.0
	A2	Wheat (shoot)	13.1	90.3	0.0200	13.5	6.5	38.7	0.0000	412.0
	A3	Strawberry (fruit)	12.1	33.2	0.0000	78.3	8.9	34.7	0.0000	112.0
	A4	Carrot (root)	22.1	25.6	0.0100	114.0	6.0	44.7	0.0100	99.4
Mean A			15.7	45.2	0.0075	56.3	9.8	41.2	0.0075	333.6
Monttar and Gaza City Center	B1	Onion (bulb)	95.0	33.2	0.0200	55.3	5.8	54.6	0.0300	447.2
	B2	Squash (fruit)	1.12	54.2	0.0300	66.0	20.1	51.2	0.0000	111.2
	B3	Spinach(leaves)	0.11	25.4	0.0000	65.2	24.5	5.80	0.0000	71.0
	B4	Cabbage(leaves)	1.21	33.2	0.0100	114.0	33.2	60.1	0.0000	115.5
Mean B			24.3	36.5	0.0150	75.1	20.9	42.9	0.0075	186.0
Al Zytoon and Shahk Ejleen	C1	Potato (tuber)	12.5	41.5	0.0400	54.0	12.2	41.2	0.0000	211.0
	C2	Carrot (root)	11.1	42.0	0.0000	88.0	14.2	55.5	0.0000	324.2
	C3	Faba bean (seed)	11.2	32.7	0.0000	22.4	22.5	65.2	0.0000	211.5
	C4	Cucumber (fruit)	10.1	30.1	0.0100	41.0	–	–	–	–
Mean C			11.1	36.7	0.0120	51.3	16.4	53.9	0.0000	248.9

concentration in all plants grown in the study area were within the normal range except for onions and carrots which accumulated Pb up to the normal concentration range.

It is clear from the results in Table 1 that the levels of Zn concentration in the studied plants grown in site A could be arranged in the following order: wheat shoot, strawberry fruit, spinach leaves and carrot root; yet, in site B it can be arranged in the following order; squash fruit, onion bulb, cabbage leaves, spinach leaves, while in site C it can be arranged in the following order: carrot root, potato tuber, faba bean seed and cucumber fruit. However, it is important to mention that the uptake of Zn depends on plant species and their plant parts. Moreover, the data revealed that leaves of some plant and root of another plant contained the highest values of Zn compared with other plants parts. It is reported that the normal Zn range in the plant is 8–400 mg kg^{-1}, while the toxic level is above 400 mg kg^{-1} (Kabata Pendias and Pendias, 1984). Therefore, it can be suggested that Zn concentration in all plants grown in the study areas was within the normal range.

It is also clear from the results in Table 1 that the levels of Fe concentration in the different parts of plants grown in site A can be arranged in the following order: carrot root, strawberry fruit, spinach leaves and wheat shoot. In site B Fe concentration level can be arranged in the following order: cabbage leaves, squash fruit, spinach leaves and onion bulb, while in site C it can be arranged in the following order: carrot root, potato tuber, cucumber fruit, faba bean seed. In this respect, Das (2000) reported that the normal iron level in plants is between 50 and 250 mg kg^{-1}. Therefore, it can be suggested that Fe concentration in all plants grown in the study areas was within the normal range.

Results in Table 1 indicated that Cd concentration in wheat shoot and carrot root of the plants grown in site A 0.02 and 0.01 mg kg^{-1}, respectively. In site B, though, Cd concentrations were 0.02, 0.03 and 0.012 mg kg^{-1} in onion bulb, squash fruit and cabbage leaves, respectively, while in site C the concentrations were 0.04 and 0.01 mg kg^{-1} for potato tuber and cucumber fruit, respectively. However, Cd could not be detected in the other plants. Kabata Pendias and Pendias (1984) have reported that the Cd concentration in contaminated plants ranges between 5 and 30 mg kg^{-1}, therefore, Cd concentration in all plants grown in the three different study sites was within the normal range.

3.2. The effect of Zn and Cd on spinach plants

The effect of heavy metals (Cd and Zn) on growth, yield and chemical composition of spinach plants as well as the

Table 2 Root length, shoot height (cm), leaf area, fresh and dry weight (g) of whole spinach plant of the second season in the two samples (30 and 60 days after sowing) as affected by different levels of Cd and Zn soil additions.

Growth character		30					60				
Plant age(days)	Treatment	Control	Zn1	Zn2	Zn3	Mean Cd	Control	Zn1	Zn2	Zn3	Mean Cd
Root length (cm)	Control	11.80	13.37	14.90	12.10	13.04	16.00	17.97	18.47	14.57	16.75
	Cd1	10.73	12.50	11.07	10.67	11.24	13.70	15.20	16.03	14.63	14.89
	Cd2	9.47	9.93	9.43	9.80	9.66	12.20	14.17	12.93	14.13	13.36
	Cd3	8.40	9.23	9.17	9.10	8.98	10.17	12.23	13.17	11.43	11.75
	Mean Zn	10.10	11.26	11.14	10.42		13.02	14.89	15.15	13.69	
	L.S.D. 0.05	Cd = 2.55Zn = NS Cd * Zn = 3.87					Cd = 1.55Zn = 1.55Cd * Zn = 2.20				
Shoot height (cm)	Control	21.2	22.3	23.9	22.2	22.4	29.6	31.60	32.7	27.0	30.2
	Cd1	18.5	16.5	19.0	19.5	18.4	25.8	28.3	25.8	27.1	26.7
	Cd2	12.9	16.9	17.7	19.4	16.7	22.9	26.2	24.9	23.3	24.4
	Cd3	12.1	14.0	16.1	15.9	14.5	15.8	24.9	23.9	21.8	21.6
	Mean Zn	16.2	17.4	19.2	19.2		23.5	27.77	26.8	24.8	
	L.S.D. 0.05	Cd = 5.60Zn = NS Cd * Zn = 7.90					Cd = 2.48Zn = 2.48 Cd * Zn = 3.53				
Leaf area	Control	325.5	374.7	429.8	334.9	366.2	1080.0	1306.3	1355.2	1135.5	1219.2
	Cd1	263.70	327.4	251.3	297.8	285.0	856.9	1221.0	1146.6	1054.5	1069.7
	Cd2	217.6	278.0	258.3	251.8	251.4	679.3	877.2	1101.1	1035.1	923.2
	Cd3	159.2	282.00	205.8	193.1	210.0	830.6	945.4	790.3	906.5	868.2
	Mean Zn	241.5	315.5	286.3	269.4		861.7	1087.5	1098.3	1032.9	
	L.S.D. 0.05	Cd = 122.1Zn = 122.1 Cd * Zn = 173.6					Cd = 136.6 Zn = 136.6 Cd * Zn = 194.3				
Fresh weight (g)	Control	14.38	14.60	14.01	12.42	13.85	33.16	36.34	37.91	31.01	34.6
	Cd1	11.96	12.74	12.36	11.81	12.22	26.35	33.75	31.79	31.59	30.87
	Cd2	10.39	12.35	10.93	10.25	10.98	21.06	27.15	28.81	24.90	25.48
	Cd3	9.13	10.80	10.06	9.67	9.91	17.36	23.68	26.62	25.16	23.20
	Mean Zn	11.46	12.62	11.84	11.04		24.48	30.23	31.28	28.16	
	L.S.D. 0.05	Cd = 3.49 Zn = NS Cd * Zn = 4.96					Cd = 3.43 Zn = 3.43 Cd * Zn = 4.89				
Dry weight (g)	Control	1.241	1.219	1.379	1.049	1.222	3.98	4.39	4.57	3.56	4.12
	Cd1	0.870	1.065	0.952	0.970	0.964	2.96	3.76	3.83	3.20	3.44
	Cd2	0.716	0.912	0.833	0.732	0.798	2.40	3.06	3.40	2.93	2.94
	Cd3	0.605	0.836	0.749	0.706	0.724	1.90	2.96	3.16	2.64	2.66
	Mean Zn	0.858	1.008	0.978	0.864		2.81	3.54	3.74	3.08	
	L.S.D. 0.05	Cd = 0.402 Zn = NS Cd * Zn = 0.572					Cd = 0.51 Zn = 0.51 Cd * Zn = 0.72				

Zn1 = 100 mg kg^{-1}, **Zn2 = 200** mg kg^{-1}, **Zn3 = 400** mg kg^{-1}. **Cd1 = 10** mg kg^{-1}, **Cd2 = 20** mg kg^{-1}, **Cd3 = 40** mg kg^{-1}.

interaction between Cd and Zn in plants, at pot experiments, were also examined. Moreover, attempts were made to reduce the toxicity effect of Cd by using different rates of Zn. Growth of spinach at the different ages (30 and 60 days after sowing) was measured by recording root length, shoot height, fresh and dry weights of the shoot, root and leaf area. At harvesting, yield and yield components were recorded. It is worthy mentioning that during the two seasons plants showed similar response to Cd and Zn soil additions, either alone or in combination.

Table 2 clearly demonstrates that Cd soil addition, either alone or combined with Zn, gradually and significantly decreased all of the studied growth characters. Reduction significantly increased when increasing the addition of Cd soil, either alone or combined with Zn, in order to reach its maximum reduction at the highest Cd level of (40 mg kg^{-1}) in the two samples of two seasons, with some exceptions being reported. Similar results have been observed for spinach (El Nabarawy, 2002) and carrot plants (Salim et al., 1992).

Data in Table 2 indicated that at the three levels of Zn all of the studied growth characters significantly increase, except for the fresh and dry weight of shoot, root and whole plant and root length, shoot height in the first sample, which were not significantly affected when compared to the control Zn-untreated plants. In this respect, Zn is an essential component of over 300 enzymes (Fox and Guerimot, 1998). In most of these enzymes, Zn constitutes an integral of the enzyme structure. It is important to mention that for both seasons, the lowest parameter values were monitored on the plants treated with the highest rate of Zn, alone or combined with Cd soil addition, as compared to the plants treated with the lowest rate of Zn alone, or plants treated with the three different rates of Cd. In this respect, Abd El Aziz et al. (1987) working on faba bean mentioned that increasing Zn soil addition from 0.5 to 1.0 kg generally reduced dry weight of shoot. It might be suggested that the favorable effect of Zn soil application on plant growth as well as the detrimental effect of high Zn soil addition might be attributed to its effect on enzymatic systems responsible for the biosynthesis of amino acid, protein, chlorophyll and photosynthesis.

Data in Table 3 reveals that, the concentration of all of the plant pigments (chlorophyll a, chlorophyll b and total carotenoids) significantly decreased with increasing Cd soil addition, either alone or combined with Zn at all levels. Concentrations of the total carotenoids in the two samples of the first season and chlorophyll b in the first sample of the first season were not significantly affected, as compared to the control untreated-Cd plants, with some exceptions. In this connection, Bazzaz and Govindjee (1974) reported that Cd adversely affects the emerge producing mechanisms of chloroplasts and mitochondria.

Moreover, all plant pigments (chlorophyll a and b) in the two samples significantly increased with increasing Zn soil addition, but chlorophyll b in the two samples of the second season which was not significantly affected, while total carotenoids in the two samples of the two seasons were not significantly affected with increasing zinc soil addition, as compared to the control untreated-Zn plant. In this respect, Garg et al. (1986) reported that the application of Zn increased chlorophyll a and b concentration.

Concerning the effect of Cd soil addition on Cd concentration in the shoots of spinach plants, it is clear from the results in Table 4 that in the two successive samples, significant and gradual increases in Cd concentrations were recorded at the spinach shoot supplied with the three rates of Cd either alone or combined with Zn when compared to the control-untreated or the control-Cd untreated plants. Cutler and Rains (1974) working on barley found three passive mechanisms for Cd uptake: exchange absorption, irreversible binding and diffusion.

Concerning the Zn concentration in the spinach shoot, the results showed that in the second sample the shoot always

Table 3 Chlorophyll a and b as well as carotenoids concentrations (mg/g fresh weight) in the leaves of spinach plant in the two samples (30 and 60 days after sowing) as affected by different levels of Cd and Zn soil additions in the second season.

Plant age (days)		30					60				
Plant Pigment	Treatment	Control	Zn1	Zn2	Zn3	Mean Cd	Control	Zn1	Zn2	Zn3	Mean Cd
Chlorophyll a	Control	3.81	3.58	2.80	3.57	3.44	5.47	5.45	4.78	5.62	5.33
	Cd1	3.20	3.31	3.15	3.38	3.26	3.58	4.17	4.22	4.36	4.08
	Cd2	2.67	2.77	2.10	3.06	2.65	3.23	4.03	4.15	4.41	3.95
	Cd3	3.04	3.06	2.75	3.04	2.97	2.93	3.87	4.22	4.38	3.85
	Mean Zn	3.18	3.18	2.70	3.26		3.80	4.38	4.34	4.69	
	L.S.D.	Cd = 0.558Zn = 0.558Cd * Zn = 0.794					Cd = 0.662Zn = 0.662 Cd * Zn = 0.941				
Chlorophyll b	Control	1.330	1.150	1.160	1.470	1.278	1.607	1.813	1.557	1.853	1.708
	Cd1	1.180	1.120	1.087	1.010	1.099	1.480	1.540	1.477	1.603	1.525
	Cd2	0.940	1.113	1.120	1.340	1.128	0.920	1.250	1.367	1.623	1.290
	Cd3	1.427	1.263	1.387	1.357	1.358	1.313	1.490	1.287	1.227	1.329
	Mean Zn	1.219	1.162	1.188	1.294		1.330	1.523	1.422	1.577	
	L.S.D. 0.05	Cd = 0.219 Zn = NS Cd * Zn = 0.311					Cd = 0.289 Zn = NS Cd * Zn = 0.411				
Carotenoids	Control	1.457	1.413	0.790	1.387	1.262	1.413	1.837	1.270	1.537	1.514
	Cd1	1.357	1.230	1.047	1.297	1.233	1.443	1.343	1.293	1.143	1.306
	Cd2	1.027	1.053	0.927	1.263	1.068	0.803	1.327	1.320	1.013	1.116
	Cd3	1.293	1.067	0.943	1.057	1.090	1.247	1.133	1.137	1.417	1.233
	Mean Zn	1.283	1.191	0.927	1.251		1.227	1.410	1.255	1.278	
	L.S.D. 0.05	Cd = NS Zn = 0.197Cd * Zn = 0.2.79					Cd = 0.228Zn = NS Cd * Zn = 0.325				

Zn1 = 100 mg kg^{-1}, **Zn2 = 200** mg kg^{-1}, **Zn3 = 400** mg kg^{-1}. **Cd1 = 10** mg kg^{-1}, **Cd2 = 20** mg kg^{-1}, **Cd3 = 40** mg kg^{-1}.

Table 4 Cd, Zn and Fe concentrations (mg kg^{-1}) of spinach shoot in the two samples (30 and 60 days after sowing) as affected by different levels of Cd and Zn soil additions in the second season.

Plants age		30					60				
(Days)	Treatment	Control	Zn1	Zn2	Zn3	Mean Cd	Control	Zn1	Zn2	Zn3	Mean Cd
Cadmium (mg kg^{-1})	Control	1.15	1.44	1.44	1.11	1.28	1.88	1.88	2.32	2.02	2.03
	Cd1	4.93	3.55	5.47	6.39	5.08	9.07	8.91	8.20	8.75	8.73
	Cd2	5.06	5.78	7.42	6.96	6.30	8.25	9.91	8.48	7.33	8.49
	Cd3	6.66	6.53	6.85	7.80	6.96	8.72	8.59	8.71	8.76	8.70
	Mean Zn	4.45	4.32	5.29	5.56		6.98	7.32	6.93	6.72	
	L.S.D. 0.05	Cd = 1.64 Zn = NS Cd * Zn = 2.34					Cd = 1.45 Zn = NS Cd * Zn = 2.06				
Zinc (mg kg^{-1})	Control	27.0	57.4	63.0	58.0	51.4	30.5	57.0	54.5	59.0	50.3
	Cd1	22.5	55.0	53.5	59.0	47.5	29.0	30.5	40.0	47.5	36.8
	Cd2	26.5	58.5	60.0	51.0	49.0	27.0	36.5	58.5	53.5	43.9
	Cd3	25.0	55.0	57.5	52.0	47.4	25.5	29.5	42.0	57.0	38.5
	Mean Zn	25.3	56.5	58.5	55.0		28.0	38.4	48.8	54.3	
	L.S.D. 0.05	Cd = 0NS Zn = 4.8 Cd * Zn = 6.7					Cd = 7.8 Zn = 7.8 Cd * Zn = 11.1				
Iron (mg kg^{-1})	Control	58.0	69.5	74.5	71.5	68.4	71.0	94.0	77.0	85.0	81.8
	Cd1	50.5	59.5	75.5	62.5	62.0	62.0	89.5	71.5	72.5	73.9
	Cd2	44.5	64.5	74.0	62.5	61.4	57.5	69.0	75.5	86.0	72.0
	Cd3	33.0	57.0	62.0	75.5	56.9	46.0	68.0	73.5	91.5	69.8
	Mean Zn	46.5	62.6	71.5	68.0		59.1	80.1	74.4	83.8	
L.S.D. 0.05		Cd = 5.8 Zn = 5.8 Cd * Zn = 8.2					Cd = 5.5 Zn = 5.5 Cd * Zn = 7.8				

Zn1 = 100 mg kg^{-1}, **Zn2 = 200** mg kg^{-1}, **Zn3 = 400** mg kg^{-1}. **Cd1 = 10** mg kg^{-1}, **Cd2 = 20** mg kg^{-1}, **Cd3 = 40** mg kg^{-1}.

contained the highest values of Cd when compared with the values in the first sample. Thus, it can be suggested that the time of explosion is a very important factor for Cd uptake and accumulation in plants. Kabata Pendias and Pendias (1984) reported that cadmium concentration in contaminated plants ranges between 5 and 30 ppm.

The results in Table 4 reveal that the range of Cd concentration in the shoots of spinach plant treated with the three different rates of Cd alone (10, 20 and 40 mg kg^{-1}) was between 2.22 and 4.93 mg kg^{-1} in the first sample and between 8.73 and 9.06 mg kg^{-1} in the second sample. Thus, it can be concluded that Cd concentrations in the shoot of spinach Cd treated plant were within the normal range and not at a toxic level. It was concluded that when Cd soil was >22.3 mg /kg the yield, nutrient content as well as physiological and biochemical properties of mulberry leaves showed distinct change, and the damage became greater as cadmium concentration increased (Chen et al., 1996).

As regards the effect of Zn soil addition to Cd concentration in spinach shoots, it is clear from the results that, there was non significant affect in the second sample but in the first sample significant decrease of Cd concentration was obtained by the shoot of the spinach plant supplied with the lowest level of Zn treatments when compared with control plants untreated Zn. In this respect, it can be suggested that Zn soil addition may be overcome to some extent, the inhibition effects of Cd on absorption, accumulation and translocation within spinach plant. Similar results were reported by Choudhary et al. (1994) on wheat plant However, Dabin et al. (1978) concluded that Zn and Cd are bound to different legends in rice roots. Also, Zn was found in some cases to depress Cd uptake indicating some kind of interaction between these two metals. On the other hand, it is important to mention that as reported before high values of Cd concentration were obtained by the shoots of spinach plants supplied with the three different rates of Zn combined with the three different rates of Cd when compared with control-untreated plants or plants supplied with the same level of Zn or Cd soil addition, in this respect, it can be suggested that the effect of Zn soil addition on plant Cd concentration might depend on the concentration of Cd and Zn in the soil.

Concerning the effect of Cd and Zn soil addition to Zn concentration in spinach plants, the results in Table 4 indicated decrease in Zn concentrations by the shoot of spinach plant supplied with the three different rates of Cd soil addition either alone or combined with Zn in the second sample alone, while no significant affect on the first sample when compared with control-untreated plant or control-Cd untreated plant. Moreover, the results revealed that the range of Zn concentration in the shoots of spinach plants treated with the three different rates of Cd alone (10, 20 and 40 mg kg^{-1}) was between 22.5 and 25.0 mg kg^{-1} in first sample and between 25.5 and 29.0 mg kg^{-1} in the second sample. As regards the effect of Zn soil addition to zinc concentration in spinach plants, the results in Table 4 revealed that significant and gradual increases in Zn concentration were recorded by shoot of spinach plant supplied with the three different rates of Zn either alone or in combined with Cd soil addition when compared with control-untreated plants or plant supplied with Cd soil addition alone. Moreover, it is clear from the results, that Zn concentration in shoot of spinach plant was in accordance to its levels in soil. It has been reported by many researchers that certain essential heavy metals, such as Zn are taken into plant cells by metabolic mechanisms (Marschner, 1995).

Regarding the Zn concentration in the shoot of spinach plants the results in Table 4, one may notice that the range of Zn concentration in the shoots of spinach plants treated with the three different rates of Zn alone (100, 200 and 400 mg kg^{-1}) was between 57.4 and 58.0 mg kg^{-1} in the first sample and between 57.0 and 59.0 mg kg^{-1} in the second sample. The normal Zn level in plant dry weight is reported to be 8–400 mg kg^{-1}, while the toxic level is >400 mg kg^{-1} (Kabata

Pendias and Pendias, 1984). Thus, zinc concentration in the spinach shoot was within the normal rang regardless of zinc soil addition levels.

Data in Table 4 also indicate gradual decreases in Fe concentrations in the shoot of spinach plant supplied with the three different rates of Cd soil addition, either alone or combined with Zn in the two samples, when compared with control-untreated plants or plants supplied with the three different rates of Cd combined with any of the three different rates of Zn. In this respect, Siedlecka and Krupa (1999) report that Cd is one of the most dangerous environmental pollutants, it interacts with Fe modifying effects of deficient or excessive Fe supply. Their distribution, plant growth and photosynthesis are explained. Also, Fe transporters have been shown to be able to transport several metals including Cd in Arabidopsis (Korshunova et al., 1999). Moreover, Yang et al. (1996) reported that influx of Fe decreased with increasing external Cd levels, in maize plant grown with Cd up to 14 μM compared to control. Thus, it can be suggested that the several detrimental effects of Cd soil addition on growth and yield of spinach plant may be partially due to decreases in Fe concentration within the plants. Regarding the effect of Zn soil addition on iron concentration in spinach shoots, results in Table 4 reveal significant increases in Fe concentration were recorded at the shoot of spinach plant, supplied with the three different rates of Zn, either alone or in combined with Cd soil addition when compared with control-untreated plants or plants supplied with Cd soil addition alone.

4. Conclusions

In conclusion, the concentration of Pb, Cd, Zn and Fe is within normal range in soils except the concentration of lead and iron in some sites of the study area. However, the accumulation of heavy metals was within the normal range and is dependent on plant species and their organs. Work is in progress to estimate toxic metal concentration in different sites of Gaza Strip–Palestine. Regarding to the effect of cadmium and zinc on spinach plants at pot experiments, the observed reduction of growth characters and plant pigments, with increasing cadmium soil addition either alone or combined with zinc soil addition might be partially due to the increase in cadmium concentration and consequently its negative effects on net photosynthesis, transpiration, chlorophyll a as well as nutrient contents.

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Journal of the Association of Arab Universities for Basic and Applied Sciences (2011) 10, 28–32

الاعتماد الخاص للذرات المتحركة ذات الانتقال ثنائي الفوتونات

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الملخص:

في هذا البحث اقترحنا طريقة مختلفة لدراسة تفاعل الذرات الباردة مع الليزر. الصيغة التحليلية لاحتمالية ا لإنبعاث الناتجة من تفاعل ذرة ثلاثية مستوى الطاقة باردة جدا مع مجال متعدد كهرومغناطيسى استنتجت في هذا البحث بالتفصيل. تم إيجاد المعادلات في حالة انتقال عديد الفوتونات عن طريق ميكانيكية الحالات المساعدة وذلك باعتبار أن الذرات تتحرك على المحور السيني داخل الحاجز والنظر لسلوك الدالة الموجية داخل الحاجز. تمت ملاحظة ظاهرة النشاط والإخماد بصورة مختلفة في هذا النموذج. ووجد أنه عندما نأخذ جزء ا لإنتشار داخل الحاجز فإن احتمالية الانبعاث للفوتونات تتأثر بشدة وهذا يعنى ضرورة أخذ هذا الجزء في الإعتبار عند دراسة هذه المسالة.

Journal of the Association of Arab Universities for Basic and Applied Sciences (2011) **10**, 28–32

University of Bahrain

Journal of the Association of Arab Universities for Basic and Applied Sciences

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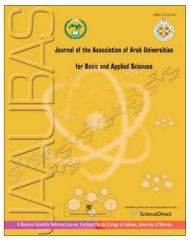

ORIGINAL ARTICLE

Spatial dependence of moving atoms with a two-photon process

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Available online 23 July 2011

KEYWORDS

Moving atoms;
Two-photon process

Abstract We propose an alternative treatment to describe the spatial-dependence of the non-resonant two-photon mazer. Previous work presented a treatment for studying the cold atom micromaser (mazer) but under certain restrictive conditions. We now extend those results to a general case taking into account the spatial dependence, off-resonant interaction and Stark-shift. In a mesa mode profile, we obtain an exactly analytic solution of the model, by means of which we analyze the analytical form of the emission probability. We demonstrate that, when the spatial dependence effects are taken into consideration, the feature of the emission probability is influenced significantly.

1. Introduction

Cold and ultracold atoms introduce new regimes in atomic physics often not considered in the past (Wieman et al., 1999). Many new concepts and new phenomena, that involve ultracold atoms, are proposed or observed. Among them are atom optics (Adams et al., 1994), Bose Einstein condensation (Griffin et al., 1995), atom lasers (Bloch et al., 1999), nonlinear atom optics (Lenz et al., 1994), nonlinear optics of matter waves (Goldstein et al., 2000) and mazer action (Scully et al., 1996; Meyer et al., 1997; Loffler et al., 1997; Schröder et al., 1997; Zhang et al., 1999; Aran et al., 2000). In these phenomena, the interaction between ultracold atoms and quantum radiation fields plays an important role (Haroche et al., 1991; Englert et al., 1991) and the quantization of the center-of-mass (c.m.) motion of atoms must be taken into account in studies on this kind of interaction. Studies in Haroche et al. (1991) and Englert et al. (1991) have been extended by including the dissipation of the photon energy into the description in Battocletti and Englert (1994). In all these previous studies, the mazer properties were always presented in the resonant case where the cavity mode frequency is equal to the atomic transition frequency. In Obada and Abdel-Aty (2000) and Abdel-Aty and Obada (2002a,b) those restrictions have been removed and a general theory of the mazer has been established. Further, the research on quantum treatment of the atomic motion has been extended to study the off-resonant case (Bastin and Martin, 2003).

On the other hand, during the last decade many theoretical and experimental efforts have been done in order to study two-photon processes involving atoms inside a cavity, stimulated by the experimental realization of a two-photon micromaser.

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Peer review under responsibility of University of Bahrain.
doi:10.1016/j.jaubas.2011.06.005

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In view of continuing technological improvements in microcavities even at optical frequencies recent work (Concannon et al., 1997) has motivated the examination of certain aspects of the two-photon mazer theory that are fundamental to the process. These aspects have their counterpart in the usual single-photon mazer, but rather different behavior is to be expected in the two-photon case, owing to the essential nonlinearity of the process. We have here in mind a degenerate two-photon mazer with the upper state connected to the lower one by a two-photon process.

We discuss the emission probability of the two-photon system, taking into account the spatio-temporal dependence. This is most conveniently accomplished in a quantum theory of the mazer formalism in terms of the dressed state approach (Scully et al., 1996; Meyer et al., 1997; Loffler et al., 1997; Schröder et al., 1997; Si-de et al., 1998, Zhang and He, 1998 and Zhang et al., 2002). To make the two-photon processes closer to the experimental realization, we include the effect of the dynamic Stark shift in the evolution of the emission probability, which is necessary and interesting. Related treatment discussing the quantum theory of the two-photon maser without the spatial dependence (i.e. in front and after leaving the cavity), have been presented in the literature (Zhang et al., 1999). However in this problem three regions are distinguished: in front of the cavity described by $\mu(z)$ where incident and reflected waves are present, inside the cavity represented by $\mu(z)$, $\mu(z, L)$ where transient regime occurs; and after leaving the cavity described by $\mu(z, L)$ where transmitted waves are present with $\mu(z)$ the step-function. Other extensions are made namely the off-resonance case and the Stark shift effect are considered. Contrary to what is claimed in Bastin and Martin, 2003 we find that the problem, in the mesa mode case, reduces to an elementary scattering problem over a potential barrier and a potential well defined by the cavity even in the presence of detuning and Stark shift.

The material of this paper is arranged as follows: in the section 'General scheme' we start with the theoretical description of the model. We obtain an exactly analytic solution of the model, by means of which we analyze the analytical form of the emission probability. Finally conclusions are presented in the last section.

2. General scheme

The concept of the mazer has been applied to the two-photon process in Zhang et al. (1999). They have studied the quantized-z-motion-induced emission and the photon statistics of the micromaser pumped by slow atoms after leaving the cavity thus they did not include terms which describe the incident and transient parts in the wave function, which when added alter the dynamics of the system. However in this problem we have three regions: one of them described by $\mu(z)$ which represents the wave function in front of the cavity, $\mu(z) - \mu(z, L)$ which represents inside the cavity of length L, and the last one with $\mu(z, L)$ which describes the wave function after leaving the cavity. But, if the ideas are to be contemplated for applications, the issue of propagation inside the cavity is crucial. Addressing this issue is the purpose of the present paper.

We consider a two-level atom moving along the z-direction in the way to a cavity of length L. The atom is coupled with a two-photon transition to a single-mode of the quantized field present in the cavity. The atom-field interaction is modulated by the cavity field mode function. The atomic center-of-mass motion is described quantum mechanically and the rotating-wave approximation is made. The Hamiltonian describing the system is given by

$$\begin{aligned}\widehat{H} = {} & \frac{\widehat{P}_z^2}{2m} + \hbar\omega(\hat{a}^\dagger\hat{a} + (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) + \hat{a}^\dagger\hat{a}(\beta_1|\uparrow\rangle \\ & \times \langle\uparrow| + \beta_2|\downarrow\rangle\langle\downarrow|)) + \frac{\hbar}{2}\Delta(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \\ & + \hbar\lambda u(z)(|\downarrow\rangle\langle\uparrow|\hat{a}^{\dagger 2} + \hat{a}^2|\uparrow\rangle\langle\downarrow|),\end{aligned} \tag{1}$$

where P_z is the atomic center-of-mass momentum along z-axis. We denote by λ, the atom-field coupling strength for the interaction between the cavity field and the atom, $u(z)$ is the mode function of the cavity field and m is the atomic mass. β_1 and β_2 are parameters describing the intensity-dependent Stark shifts of the two levels that are due to the virtual transitions to the intermediate relay. The operator a (a^+) is the annihilation (creation) operator for the cavity field, ω is the field frequency and Δ the detuning parameter. When a cold atom is approaching the interaction region, it can be reflected or transmitted according to quantum scattering theory.

In the interaction picture, let us write Eq. (1) in the following form:

$$\begin{aligned}\widehat{H} &= \frac{\widehat{P}_z^2}{2m} + \widehat{V}, \\ \widehat{V} &= \beta_1(|\uparrow\rangle\langle\uparrow| + \beta_2\hat{a}^\dagger\hat{a}|\downarrow\rangle\langle\downarrow|) + \frac{\hbar}{2}\Delta(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \\ &\quad + \hbar\lambda u(\hat{z})(|\downarrow\rangle\langle\uparrow|\hat{a}^{\dagger 2} + \hat{a}^2|\uparrow\rangle\langle\downarrow|).\end{aligned} \tag{2}$$

The global Hilbert space of the system is given by $H = H_z \otimes H_A \otimes H_R$ with H_z the space of the wave functions describing the one-dimensional atomic center-of-mass motion, H_A the space describing the atomic internal degree of freedom, and H_R the space of the cavity single mode radiation. It is expedient to expand the atom-field state in terms of the states

$$\begin{aligned}|\Phi_0\rangle &= |0, \downarrow\rangle, \\ |\Phi_1\rangle &= |1, \downarrow\rangle, \\ |\Phi_n^+\rangle &= \sin\left(\frac{\varrho_n}{2}\right)|n, \uparrow\rangle + \cos\left(\frac{\varrho_n}{2}\right)|n+2, \downarrow\rangle, \\ |\Phi_n^-\rangle &= \cos\left(\frac{\varrho_n}{2}\right)|n, \uparrow\rangle - \sin\left(\frac{\varrho_n}{2}\right)|n+2, \downarrow\rangle,\end{aligned} \tag{3}$$

where

$$\begin{aligned}\varrho_n &= 2 \times \tan^{-1}\left(\frac{\lambda u(z)\sqrt{(n+1)(n+2)}}{\mu_n - \left(\frac{\Delta}{2} + \frac{1}{2}[n\beta_2 - (n+2)\beta_1]\right)}\right), \\ \mu_n &= \sqrt{\left(\frac{\Delta}{2} + \frac{1}{2}[n\beta_2 - (n+2)\beta_1]\right)^2 + \lambda^2 u^2(z)(n+1)(n+2)}.\end{aligned} \tag{4}$$

The states $|\Phi_n^\pm\rangle$ are z-dependent through the trigonometric functions, they satisfy

$$\begin{aligned}\frac{\partial}{\partial z}|\Phi_n^\pm\rangle &= \pm|\Phi_n^\mp\rangle\frac{d\varrho_n}{dz}, \\ \frac{\partial^2}{\partial z^2}|\Phi_n^\pm\rangle &= \pm|\Phi_n^\mp\rangle\frac{d^2\varrho_n}{dz^2} - |\Phi_n^\pm\rangle\left(\frac{d\varrho_n}{dz}\right)^2.\end{aligned} \tag{5}$$

Then, $\langle z|\Psi(t)\rangle$ can be expanded in the form $\langle z|\Psi(t)\rangle = \sum_n c_n^\pm(z, t)|\Phi_n^\pm\rangle$ and it satisfies the Schrödinger equation

$$i\frac{\partial}{\partial t}|\langle z|\Psi(t)\rangle = \widehat{H}\langle z|\Psi(t)\rangle.$$

Hence the coefficients $C_n^{\pm}(z,t)$ satisfy the coupled equation

$$\frac{\partial C_n^{+}(z,t)}{\partial t} = \left(-\frac{1}{2m}\frac{\partial^2}{\partial z^2} + V_n^{+} - \left(\frac{d\varrho_n}{dz}\right)^2\right)C_n^{+}(z,t) - \left(2\frac{C_n^{-}(z,t)}{\partial z}\left(\frac{d\varrho_n}{dz}\right) + C_n^{-}(z,t)\left(\frac{d\varrho_n}{dz}\right)^2\right), \tag{6}$$

$$\frac{\partial C_n^{-}(z,t)}{\partial t} = \left(-\frac{1}{2m}\frac{\partial^2}{\partial z^2} + V_n^{-} - \left(\frac{d\varrho_n}{dz}\right)^2\right)C_n^{-}(z,t) + \left(2\frac{C_n^{+}(z,t)}{\partial z}\left(\frac{d\varrho_n}{dz}\right) + C_n^{+}(z,t)\left(\frac{d\varrho_n}{dz}\right)^2\right), \tag{7}$$

where

$$V_n^{\pm}(z) = \frac{\hbar}{2}[n\beta_2 + (n+2)\beta_1] \pm \hbar\sqrt{\left(\frac{\Delta}{2} + \frac{1}{2}[n\beta_2 - (n+2)\beta_1]\right)^2 + \lambda^2 u^2(\hat{z})(n+1)(n+2)}, \tag{8}$$

Eqs. (6) and (7) mean that, we get for each n two coupled partial differential equations. But once $u(z)$ is taken to be constant, then $\frac{d\varrho_n}{dz}$ will vanish and these equations are decoupled over the entire z-axis and the problem reduces to an elementary scattering problem over a potential barrier and a potential well defined by the cavity even in the presence of detuning and Stark shift. This can be contrasted with what has been claimed earlier (Bastin and Martin, 2003).

In what follows we study the mesa-mode case of the field which means that $u(z)$ is constant inside the cavity and zero outside the cavity. In this case the states (3) become the dressed states of the system. We assume that, initially, the atomic center-of-mass motion is not correlated to the other degrees of freedom. We describe it by the wave packet

$$X(z) = \langle z|\phi(0)\rangle = \int dk G(k) e^{ikz}\theta(-z). \tag{9}$$

We denote by $\theta(-z)$ the Heaviside step function (indicating that the atoms are incident from the left of the cavity). The Fourier amplitudes $G(k)$ are adjusted such that the center of the wave packet enters the cavity at time $t = 0$. The initial state of the system can be written

$$\begin{aligned}|\psi(0)\rangle &= \sum_{n=0}^{\infty} q_n|n\rangle\left[\cos\left(\frac{\tau}{2}\right)|\uparrow\rangle + \sin\left(\frac{\tau}{2}\right)e^{-i\varphi}|\downarrow\rangle\right]\\ &= \sin\left(\frac{\tau}{2}\right)e^{-i\varphi}(q_0|0,\downarrow\rangle + q_1|1,\downarrow\rangle)\\ &\quad + \sum_{n=0}^{\infty}\left\{q_n\cos\left(\frac{\tau}{2}\right)|n,\uparrow\rangle + q_{n+2}\sin\left(\frac{\tau}{2}\right)e^{-i\varphi}|n+2,\downarrow\rangle\right\}\\ &= \sin\left(\frac{\tau}{2}\right)e^{-i\varphi}(q_0|\Phi_0\rangle + q_1|\Phi_1\rangle) + \sum_{n=0}^{\infty}(Y_{n-}|\Phi_n^{-}\rangle + Y_{n+}|\Phi_n^{+}\rangle),\end{aligned} \tag{10}$$

where

$$\begin{aligned}Y_{n+} &= q_n\sin\left(\frac{\varrho n}{2}\right)\cos\left(\frac{\tau}{2}\right) + q_{n+2}\cos\left(\frac{\varrho n}{2}\right)\sin\left(\frac{\tau}{2}\right)e^{-i\varphi},\\ Y_{n-} &= q_n\cos\left(\frac{\varrho n}{2}\right)\cos\left(\frac{\tau}{2}\right) - q_{n+2}\sin\left(\frac{\varrho n}{2}\right)\sin\left(\frac{\tau}{2}\right)e^{-i\varphi}.\end{aligned} \tag{11}$$

In a mesa mode profile, the wave function of the atom-field interaction can be obtained using a straightforward calculation, in the following form

$$\begin{aligned}\langle z|\Psi(t)\rangle = \int dk G(k)\exp\left(-i\frac{\hbar k^2 t}{2M}\right)\sum_{\infty}^{n=0} q_n\\ \times\Big[\Big(\Big\{e^{ikz} + \Big(A_n^{+}\sin\left(\frac{\varrho n}{2}\right)\cos\left(\frac{\tau}{2}\right)\\ + A_n^{-}\cos\left(\frac{\varrho n}{2}\right)\sin\left(\frac{\tau}{2}\right)e^{-i\varphi}\Big)e^{-ikz}\Big\}\theta(-z)\\ + \Big\{B_n^{+}\sin\left(\frac{\varrho n}{2}\right)\cos\left(\frac{\tau}{2}\right) + B_n^{-}\cos\left(\frac{\varrho n}{2}\right)\sin\left(\frac{\tau}{2}\right)e^{-i\varphi}\Big\}\\ \times e^{ik(z-L)}\theta(z-L)\\ + \Big\{\Big(\alpha_n^{+}e^{ik_n^{+}z} + \beta_n^{+}\times e^{-ik_n^{+}z}\Big)\sin\left(\frac{\varrho n}{2}\right)\cos\left(\frac{\tau}{2}\right)\\ \times\left(\alpha_n^{-}e^{ik_n^{-}z} + \beta_n^{-}e^{-ik_n^{-}z}\right)\cos\left(\frac{\varrho n}{2}\right)\sin\left(\frac{\tau}{2}\right)\times e^{i\varphi}\Big\}\\ \times[\theta(z) - \theta(z-l)])|n,\uparrow\rangle\\ + \Big(\Big\{A_n^{+}\cos\left(\frac{\varrho n}{2}\right)\sin\left(\frac{\tau}{2}\right)e^{-i\varphi} - A_n^{-}\sin\left(\frac{\varrho n}{2}\right)\cos\left(\frac{\tau}{2}\right)\Big\}\\ \times e^{-ikz}\theta(-z) + \Big\{B_n^{+}\cos\left(\frac{\varrho n}{2}\right)\sin\left(\frac{\tau}{2}\right)e^{-i\varphi} - B_n^{-}\sin\left(\frac{\varrho n}{2}\right)\\ \times\cos\left(\frac{\tau}{2}\right)\Big\}e^{ik(z-L)}\theta(z-L)\\ + \Big\{\Big(\alpha_n^{+}e^{ik_n^{+}z} + \beta_n^{+}e^{-ik_n^{+}z}\Big)\cos\left(\frac{\varrho n}{2}\right)\sin\left(\frac{\tau}{2}\right)\\ \times e^{-i\varphi}\left(\alpha_n^{-}e^{ik_n^{-}z} - \beta_n^{-}e^{-ik_n^{-}z}\right)\times\sin\left(\frac{\varrho n}{2}\right)\cos\left(\frac{\tau}{2}\right)\Big\}\\ \times[\theta(z) - \theta(z-L)])|n+2,\downarrow\rangle],\end{aligned} \tag{12}$$

where, the coefficients $A_n^{\pm}$, of the reflected waves $B_n^{\pm}$, of the transmitted waves and $\alpha_n^{\pm}$ and $\beta_n^{\pm}$ of the transient regime are given by

$$\begin{aligned}A_n^{\pm}(k) &= \frac{i\Upsilon_n^{\pm}\sin(k_n^{\pm}L)}{\cos(k_n^{\pm}L) - i\delta_n^{\pm}\sin(k_n^{\pm}L)}, \quad B_n^{\pm}(k) = \frac{e^{-ikL}}{\cos(k_n^{\pm}L) - i\delta_n^{\pm}\sin(k_n^{\pm}L)},\\ \alpha_n^{\pm}(k) &= \frac{\frac{1}{2}\left(1 + \frac{k}{k_n^{\pm}}\right)e^{-ik_n^{\pm}L}e^{-ikL}}{\cos(k_n^{\pm}L) - i\delta_n^{\pm}\sin(k_n^{\pm}L)}, \quad \beta_n^{\pm}(k) = \frac{\frac{1}{2}\left(1 - \frac{k}{k_n^{\pm}}\right)e^{ik_n^{\pm}L}e^{-ikL}}{\cos(k_n^{\pm}L) - i\delta_n^{\pm}\sin(k_n^{\pm}L)},\end{aligned} \tag{13}$$

where

$$\begin{aligned}&\delta_n^{\pm} = \frac{1}{2}\left(\frac{k_n^{\pm}}{k} + \frac{k}{k_n^{\pm}}\right), \quad \Upsilon_n^{\pm} = \frac{1}{2}\left(\frac{k_n^{\pm}}{k} - \frac{k}{k_n^{\pm}}\right),\\ &k_0^2 = \sqrt{k^2 + \frac{m\Delta}{\hbar}}, \quad k_1^2 = \sqrt{k^2 + \frac{m}{\hbar}(\Delta - 2\beta_2)},\\ &k_n^{\pm} = \sqrt{k^2 - \frac{\gamma^2}{2}[n\beta_2 - (n+2)\beta_1] \mp \gamma^2\sqrt{\Delta^2/\lambda^2 + (n+1)(n+2)}}.\end{aligned} \tag{14}$$

It is to be noted that the vacuum coupling energy $\hbar\lambda = (\hbar\gamma)^2/2m$, and $\hbar k$ is the atomic center-of-mass momentum. To make a shortcut to the two-photon JC-model considered in the standard studies of the quantum optics we may write the time dependent exponent $\exp[it\hbar k^2/2m]$ in the term with $[\theta(z) - \theta(z-l)]$, i.e., inside the cavity in the following form $\exp\left(-i\left[\frac{\hbar k_n^{\pm}}{2m} \pm E_n\right]t\right)$, where $E_n = \frac{1}{2}[n\beta_2 + (n+2)\beta_1] + \hbar\sqrt{\Delta^2 + \lambda^2(n+1)(n+2)}$.

When the spatial dependence is not taken into consideration, the wave function goes automatically to the well known wave function for the standard two-photon JC-model. The

solution (10) contains the regions inside and outside the cavity. The region inside the cavity, the contributions of the dynamic Stark effect and the off-resonant case have not been considered in earlier studies for the two-photon cases (Zhang et al., 1999).

3. Emission spectra

If the cavity field is initially prepared in the coherent state, we have the following photon-number distribution

$$P(n) = \exp(\bar{n})\frac{\bar{n}^n}{n!} = q_n^2, \tag{15}$$

where $\bar{n}$ is the averaged photon number. With the wave function calculated, any property related to the atom or the field can be calculated. Let us denote by $\rho(t)$ the atom-field density matrix, that its elements ρ_{ij} are

$$\rho_{ij} = \sum_n \langle i, z|\psi(t)\rangle\langle\psi(t)|z, j\rangle. \tag{16}$$

With the wave function calculated, any property related to the atom or the field can be calculated.

$$\rho_{ij} = \sum_n |\langle i, z|\psi(t)\rangle|^2. \tag{17}$$

The probability of finding the atom in the upper state is given by $\rho_{ee} = \langle C|C\rangle$ and the probability of being in the ground state is given by $\rho_{gg} = \langle S|S\rangle$, where

$$\begin{aligned}|C\rangle = \int dk G(k) \exp\left(-i\frac{\hbar k^2 t}{2M}\right) \sum_n qn \\ \times \left(\left[e^{ikz} + \left(A_n^+ \sin\theta_n + A_n^- \cos\theta_n\right) \times e^{-ikz}\right]\theta(-z)\right. \\ + \left(B_n^+ \sin\theta_n + B_n^- \cos\theta_n\right) e^{ik(z-L)}\theta(z-L) \\ + \left\{\left(\alpha_n^+ \times e^{ik_n^+ z} + \beta_n^+ e^{-ik_n^+ z}\right)\sin\theta_n\right. \\ \left.\left. + \left(\alpha_n^- e^{ik_n^- z} + \beta_n^- e^{-ik_n^- z}\right)\cos\theta_n\right\}[\theta(z) - \theta(z-L)]\right)|n\rangle,\end{aligned} \tag{18}$$

$$\begin{aligned}|S\rangle = \int dk G(k) \exp\left(-i\frac{\hbar k^2 t}{2M}\right) \sum_n qn\left(\left(A_n^+ \cos\theta_n - A_n^- \sin\theta_n\right)e^{-ikz}\right. \\ \times\theta(-z) + \left(B_n^+ \cos\theta_n - B_n^- \sin\theta_n\right)e^{ik(z-L)}\theta(z-L) \\ + \left\{\left(\alpha_n^+ e^{ik_n^+ z} + \beta_n^+ e^{-ik_n^+ z}\right)\cos\theta_n - \left(\alpha_n^- e^{ik_n^- z} - \beta_n^- e^{-ik_n^- z}\right)\sin\theta_n\right\} \\ \left.\times[\theta(z) - \theta(z-L)]\right)|n+2\rangle.\end{aligned} \tag{19}$$

In what follow, we shall investigate the properties of the spatial dependence on the emission probability when we take the distribution function $G(k) = \delta(k - k_0)$. The emission probability is given by

$$\begin{aligned}P_s = \sum_n P(n)\left|\left\{A_n^+ \cos\theta_n - A_n^- \sin\theta_n\right\}\right|^2 \\ + \left|\left\{B_n^+ \cos\theta_n - B_n^- \sin\theta_n\right\}\right|^2 \\ + \left|\left\{\left(\alpha_n^+ e^{ik_n^+ z} + \beta_n^+ e^{-ik_n^+ z}\right)\cos\theta_n - \left(\alpha_n^- e^{ik_n^- z} - \beta_n^- e^{-ik_n^- z}\right)\sin\theta_n\right\}\right|^2.\end{aligned} \tag{20}$$

The spatial dependence of the cavity field shows that the emission probability depends not only on the statistics of the field but also on the momentum distribution of the atomic mass center. The first part in Eq. (20) is the contribution from the reflected waves, the second part is due to the transmitted waves, and the third part is the contribution of the transient regime.

This last part plays an essential role in the emission probability. It does not appear possible to express the sums in Eq. (20) in closed form, but for not too large mean photon number, direct numerical evolutions can be performed. Resorting to Eq. (20), in the following we investigate the response of the atom to the coherent cavity field when it experiences a transition from classical regime to the quantum regime.

4. Conclusions

In this paper we have presented the non-resonant two-photon mazer in the presence of Stark shift effect taking into account the spatial-dependence. The full solution is given and the case of the inter-cavity is considered in detail. The situation here is somewhat different from the cold atom scheme that has already been examined (Scully et al., 1996; Meyer et al., 1997; Loffler et al., 1997; Schröder et al., 1997). The emission probability of the system is also calculated and investigated with special emphasis on its spatial dependence. The results of this paper may be tested with micromaser-like experiments by using a high-Q micromaser pumped by cold atoms with very high principal quantum number (Walther et al., 2000; Varcoe et al., 2000; Weidinger et al., 1999).

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Journal of the Association of Arab Universities for Basic and Applied Sciences (2011) 10, 33–41

القيم الشبكية للبنيات المزدوجه المتناسقة

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الملخص:

فى هذا البحث تم تقديم مفهوم البنيات المزدوجه المتناسقة ذات القيم الشبكية . وأيضا تمت دراسة العلاقة الطبيعية بين البنيات ال مزدوجه المتناسقة والفضاءات التوبولوجية المزدوجه والبنيات التقاربية المنتظمة المزدوجه ودراسة خواص هذه التركيبات.

Journal of the Association of Arab Universities for Basic and Applied Sciences (2011) **10**, 33–41

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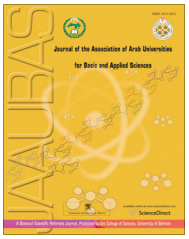

ORIGINAL ARTICLE

Lattice valued double syntopogenous structures

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Available online 2 August 2011

KEYWORDS

Double fuzzy topological spaces;
Topogenous structure;
Quasi-proximity

Abstract In this paper, we introduce the concept of lattice valued double fuzzy syntopogenous structures in framework of double fuzzy topology (proximity and uniformity). Some fundamental properties of them are established. Finally, a natural links between double fuzzy syntopogenous structure, double fuzzy topology, double fuzzy proximity and double fuzzy uniformity are given.

1. Introduction and preliminaries

Kubiak (1985) and Šostak (1985) introduced the notion of (L-) fuzzy topological space as a generalization of L-topological spaces (originally called (L-)fuzzy topological spaces by Chang (1968) and Goguen (1973). It is the grade of openness of an L-fuzzy set. A general approach to the study of topological-type structures on fuzzy powersets was developed in Höhle (1980), Höhle and Šostak (1995), and Kubiak (1985).

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov (1986). Recently, Çoker (1997) and Çoker et al., 1996 introduced the notion of intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Samanta and Mondal (2002) introduced the notion of intuitionistic gradation of openness which a generalization of both of L-fuzzy topological spaces and the topology of intuitionistic fuzzy sets (Çoker, 1997).

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Peer review under responsibility of University of Bahrain.
doi:10.1016/j.jaubas.2011.06.007

Working under the name "intuitionistic" did not continue because doubts were thrown about the suitability of this term, especially when working in the case of complete lattice L. These doubts were quickly ended in 2005 by Garcia and Rodabaugh, 2005. They proved that this term is unsuitable in mathematics and applications. They concluded that they work under the name "double".

Csaszar (1963) gave a new method for the foundation of general topology based on the theory of syntopogenous structure to develop a unified approach to the three main structures of set-theoretic topology: topologies, uniformities and proximities. This enabled him to evolve a theory including the foundations of the three classical theories of topological spaces, uniform spaces and proximity spaces. In the case of the fuzzy structures there are at least two notions of fuzzy syntopogenous structures, the first notion worked out in (Katsaras, 1988, 1985a, 1983) presents a unified approach to the theories of Chang (1968) fuzzy topological spaces, Hutton fuzzy uniform spaces (Hutton, 1977), Katsaras fuzzy proximity spaces (Katsaras, 1985b, 1980, 1979) and Artico fuzzy proximity (Artico and Moresco, 1984). The second notion worked out in Katsaras (1991, 1990) agree very well with Lowen fuzzy topological spaces (Lowen, 1976), Lowen–Hohle fuzzy uniform spaces (Lowen, 1981) and Artico–Moresco fuzzy proximity spaces (Artico and Moresco, 1984).

In this paper, we establish the concept of double fuzzy syntopogenous structures as a unified approach to theores of (Hohle and Rodabaugh) double fuzzy topology, double fuzzy proximity spaces and double fuzzy uniformity spaces. Some

fundamental properties of them are established. Finally, the relationship among double fuzzy syntopogenous structures, double fuzzy topology, double fuzzy proximity and double fuzzy uniformity is studied.

Throughout this paper, let X be a nonempty set and $L = (L, \leqslant, \vee, \wedge, \bot, \top)$ a completely distributive lattice where $\bot(\top)$ denotes the universal lower (upper) bound.

Definition 1.1. CQML, the category of **complete quasi-monoidal lattices**, (Rodabaugh, 2003).

Comprises the following data, where composition and identities are taken from **SET:**

(1) **Objects**: $(L, \leqslant, \odot)$ where $\odot : L \times L \to L$ is isotone and $\top \odot \top = \top$.
(2) **Morphisms:** All **SET** morphisms preserves $\odot$, $\top$ and arbitrary $\vee$.

Definition 1.2. Categories related to **CQML** (Rodabaugh, 2003).

(1) **QUML**, the category of quasi-uniform monoidal lattices is the full subcategory of **CQML** for which $\leftarrow \odot \leftarrow$ is associative, commutative and $\leftarrow \top \leftarrow$ is identity.
(2) **DQML**, the category of deMorgan quasi-monoidal lattices is the full subcategory of **CQML** for which * is an order-reversing involution and each morphism preserves the involution.
(3) **QUANT**, the category of **quantales** is the full subcategory of **CQML** for which $\odot$ is distributive over arbitrary joins, i.e.,

$$(\underset{i\in\Gamma}{\vee} r_i) \odot s = \underset{i\in\Gamma}{\vee}(r_i \odot s).$$

(4) **CQUANT**, the category of **coquantales** is the full subcategory of **CQML** for which $\odot$ is distributive over arbitrary meets, i.e.,

$$(\underset{i\in\Gamma}{\wedge} r_i) \odot s = \underset{i\in\Gamma}{\wedge}(r_i \odot s).$$

(5) **DQUAT,** the category of **deMorgan**, **quasi-uniform monoidal quantales.**
In this paper, for each $(L, \leqslant, \odot, *) \in$ **DQUAT**, we define $x \oplus y = (x^* \odot y^*)^*$.
(6) **DBIQUAT = DQUAT** $\cap$ **T COQUANT.**
(7) **CMVAL**, the category of complete MV-algebra is the full subcategory of **DBIQUAT** for which it satisfies

(MV) $(x \mapsto y) \mapsto y = x \vee y$, for all x, $y \in L$ where $x \mapsto y$ is defined by $x \mapsto y = \vee \{z \mid x \odot z \leqslant y\}$ and $x^* = x \mapsto \bot$.

Definition 1.3 (Kim and Ko, 2008). Let $(L, \leqslant, \odot, \oplus, *) \in |\,\mathbf{DQUAT}\,|$ and $\phi : X \to Y$ be a function. For each $x, \{y, z \in L, y_i \mid i \in \Gamma, f, g \in L^X$ and $f_i \in L^Y$. we have:

(1) If $y \leqslant z$, $(x \odot y) \leqslant (x \odot z)$ and $(x \oplus y) \leqslant (x \odot z)$.
(2) $x \odot y \leqslant x \wedge y \leqslant x \vee y \leqslant x \oplus y$.
(3) $\underset{i\in\Gamma}{\wedge} y_i^* = (\underset{i\in\Gamma}{\vee} y_i)^*$ and $\underset{i\in\Gamma}{\vee} y_i^* = (\underset{i\in\Gamma}{\wedge} y_i)^*$.
(4) $x \oplus (\underset{i\in\Gamma}{\wedge} y_i) = \underset{i\in\Gamma}{\wedge}(x \oplus y_i)$.

Definition 1.4 (Cetkin and Aygun, 2010). The maps $\mathcal{T}, \mathcal{T}^* : L^X \to L$ is called double fuzzy topology on X if it satisfies the following conditions:

$(LO1)$ $\mathcal{T}(f) \leqslant (\mathcal{T}^*(f))^*$, for all $f \in L^X$,
$(LO2)$ $\mathcal{T}(1_\emptyset) = \mathcal{T}(1_X) = \top$ and $\mathcal{T}^*(1_\emptyset) = \mathcal{T}^*(1_X) = \bot$,
$(LO3)$ $\mathcal{T}(f_1 \odot f_2) \geqslant \mathcal{T}(f_1) \odot \mathcal{T}(f_1)$ and $\mathcal{T}^*(f_1 \odot f_2) \leqslant \mathcal{T}^*(f_1) \oplus \mathcal{T}^*(f_1)$, for each f_1, $f_2 \in L^X$,
$(LO4)$ $\mathcal{T}(\underset{i\in\Delta}{\vee} f_i) \geqslant \underset{i\in\Delta}{\wedge} \mathcal{T}(f_i)$ and $\mathcal{T}^*(\underset{i\in\Delta}{\vee} f_i) \leqslant \underset{i\in\Delta}{\vee} \mathcal{T}^*(f_i)$ for each $f_i \in L^X$, $i \in \Delta$.

The pair $(X, \mathcal{T}, \mathcal{T}^*)$ is called an double fuzzy topological space (dfts, for short).

Let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be dfts's. A map $\phi : X \to Y$ is called fuzzy continuons iff $\mathcal{T}_2 \leqslant \mathcal{T}_1 \circ \phi_L^{\leftarrow}$ and $\mathcal{T}_2^* \geqslant \mathcal{T}_1^* \circ \phi_L^{\leftarrow}$.

2. Double fuzzy topogenous order and double fuzzy topologies

Definition 2.1. A maps η, $\eta^* : L^X \times L^X \to L$ is called double fuzzy semi-topogenous order on X if it satisfies the following axioms:

$(LST1)$ $\eta(f, g) \leqslant (\eta^*(f, g))^*$, for all f, $g \in L^X$,
$(LST2)$ $\eta(1_X, 1_X) = \eta(1_\emptyset, 1_\emptyset) = \top$ and $\eta^*(1_X, 1_X) = \eta^*(1_\emptyset, 1_\emptyset) = \bot$,
$(LST3)$ If $\eta(f, g) \neq \bot$ and $\eta^*(f, g) \neq \top$, then $f \leqslant g$,
$(LST4)$ If $f_1 \leqslant f$, $g_1 \leqslant g$, then $\eta(f_1, g_1) \leqslant \eta(f, g)$ and $\eta^*(f_1, g_1) \geqslant \eta^*(f, g)$.

Proposition 2.2. *Let (η, η^*) be a double fuzzy semi-topogenous order on X and let the mappings η^s, $\eta^{s*} : L^X \times L^X \to L$ defined by $\eta^s(f, g) = \eta(g^*, f^*)\psi$ and $\eta^{s^*}(f, g) = \eta^*(g^*, f^*)$, $\forall f$, $g \in L^X$. Then (η^s, η^{s*}) is double fuzzy semi-topogenous order on X.*

Definition 2.3. A double fuzzy semi-topogenous order (η, η^*) is called symmetric if $(\eta, \eta^*) = (\eta^s, \eta^{s*})$, that is,

(LST4) $\eta(f, g) = \eta(g^*, f^*)$ and $\eta^*(f, g) = \eta^*(g^*, f^*)$, $\forall f, g \in L^X$.

Definition 2.4. A double fuzzy semi-topogenous order (η, η^*) is called double fuzzy topogenous if for any f, f_1, f_2, g, g_1, $g_2 \in L^X$.

(LST5) $\eta(f_1 \oplus f_2, g) \geqslant \eta(f_1, g) \odot \eta(f_2, g)$ and $\eta^*(f_1 \oplus f_2, g) \leqslant \eta^*(f_1, g) \oplus \eta^*(f_2, g)$,
(LST6) $\eta(f, g_1 \odot g_2) \geqslant \eta(f, g_1) \odot \eta(f, g_2)$ and $\eta^*(f, g_1 \odot g_2) \leqslant \eta^*(f, g_1) \oplus \eta^*(f, g_2)$.

Definition 2.5. A double fuzzy semi-topogenous order (η, η^*) is called perfect if (LST7) $\eta(\underset{i\in\Gamma}{\vee} f_i, g) \geqslant \underset{i\in\Gamma}{\wedge} \eta(f_i, g)$ and $\eta^*(\underset{i\in\Gamma}{\vee} f_i, g) \leqslant \underset{i\in\Gamma}{\vee} \eta^*(f_i, g)$, for any $\{g, f_i : i \in \Gamma\} \subset L^X$.

An perfect double fuzzy semi-topogenous order (η, η^*) is called biperfect if (LST8) $\eta(f, \underset{i\in\Gamma}{\wedge} g_i) \geqslant \underset{i\in\Gamma}{\wedge} \eta(f, g_i)$ and $\eta^*(f, \underset{i\in\Gamma}{\wedge} g_i) \leqslant \underset{i\in\Gamma}{\vee} \eta^*(f, g_i)$, for any $\{g, f_i : i \in \Gamma\} \subset L^X$.

Theorem 2.6. *Let (η_1, η_1^*) and (η_2, η_2^*) be perfect (respectively, double fuzzy topogenous, biperfect) double fuzzy semi-topogenous order on X. Define the compositions $\eta_1 \circ \eta_2$ of η_1 and η_2 and $\eta_1^* \circ \eta_2^*$ of η_1^* and η_2^* on X by $\eta_1 \circ \eta_2(f, g) = \underset{h\in L^X}{\vee}[\eta_1(f, h) \odot \eta_2(h, g)]$ and $\eta_1^* \circ \eta_2^*(f, g) = \underset{h\in L^X}{\wedge}[\eta_1^*(f, h) \oplus \eta_2^*(h, g)]$.*

Then $(\eta_1 \circ \eta_2, \eta_1^* \circ \eta_2^*)$is perfect (resp. double fuzzy topogenous, biperfect) double fuzzy semi- topogenous order on X.

Proof. Let (η_1, η_1^*) and (η_2, η_2^*) be perfect double fuzzy semi-topogenous order on X. Then (LST3) If $\eta_1 \circ \eta_2(f,g) \neq \perp$ and $\eta_1^* \circ \eta_2^*(f,g) \neq \top$ then there exists $h \in L^X$ such that $\eta_1 \circ \eta_2(f,g) \geqslant \eta_1(f,h) \odot \eta_2(h,g) \neq \perp$ and $\eta_1^* \circ \eta_2^*(f,g) \leqslant \eta_1^*(f,h) \oplus \eta_2^*(h,g) \neq \top$. It implies $f \leqslant h \leqslant g$.

(LST7) It is proved from:

$$\begin{aligned}\eta_1 \circ \eta_2(\underset{i\in\Gamma}{\vee} f_i, g) &= \underset{h\in L^X}{\vee}\left[\eta_1(\underset{i\in\Gamma}{\vee} f_i, h) \odot \eta_2(h,g)\right] \\ &\geqslant \underset{i\in\Gamma}{\wedge}\left[\underset{h\in L^X}{\vee}[\eta_1(f_i,h) \odot \eta_2(h,g)]\right] \\ &= \underset{i\in\Gamma}{\wedge} \eta_1 \circ \eta_2(f_i, g),\end{aligned}$$

$$\begin{aligned}\eta_1^* \circ \eta_2^*(\underset{i\in\Gamma}{\vee} f_i, g) &= \underset{h\in L^X}{\wedge}\left[\eta_1^*(\underset{i\in\Gamma}{\vee} f_i, h) \oplus \eta_2^*(h,g)\right] \\ &\leqslant \underset{i\in\Gamma}{\vee}\left[\underset{h\in L^X}{\wedge}[\eta_1^*(f_i,h) \oplus \eta_1^*(h,g)]\right] \\ &= \underset{i\in\Gamma}{\vee} \eta_1^* \circ \eta_2^*(f_i, g).\end{aligned}$$

Others are easily proved. □

Definition 2.7. A double fuzzy syntopogenous structure on $X\Psi$ is a non-empty family $\Upsilon_X\Psi$ of double fuzzy topogenous orders on X satisfying the following two conditions:

(LS1) $\Upsilon_X\Psi$ is directed, i.e. given two double fuzzy topogenous orders (η_1, η_1^*), $(\eta_2, \eta_2^*) \in \Upsilon_X$, there exists a double fuzzy topogenous order $(\eta_1, \eta^*) \in \Upsilon_X$ such that $\eta \geqslant \eta_1$, η_2 and $\eta^* \leqslant \eta_1^*$, η_2^*.

(LS2) For every $(\eta, \eta^*) \in \Upsilon_X$, there exists$(\eta_1^{,}\eta_1^*) \in \Upsilon_X$, such that $\eta \leqslant \eta_1 \circ \eta_2$and $\eta^* \geqslant \eta_1^* \circ \eta_2^*$.

Definition 2.8. A double fuzzy syntopogenous structure $\Upsilon_X\Psi$ is called double fuzzy topogenous if $\Upsilon_X\Psi$ consists of a single element. In this case, $\Upsilon_X\Psi = \{(\eta, \eta^*)\}$ is called a double fuzzy topogenous structure, denoted by $\Upsilon_X\Psi = \{(\eta, \eta^*)\} = (\eta, \eta^*)$ and (X, Υ_X) is called double fuzzy topogenous space.

A double fuzzy syntopogenous structure $\Upsilon_X\Psi$ is called perfect (resp. biperfect, symmetric) if each double fuzzy topogenous order$(\eta, \eta^*) \in \Upsilon_X$ is perfect (resp.respect, biperfect, symmetric).

Theorem 2.9. *Let (η, η^*) topogenous order on X. The mapping $C_\eta : L^X \times L_0 \times L_1 \to L^X$, is defined by*

$$C_{\eta,\mu^*}(f,r,s) = \wedge\{g^* \in L^X : \eta(g,f^*) > r \quad \text{and } \eta^*(g,f^*) \leqslant s\}.$$

For each $f, f_1, f_2 \in L^X$ and $r, r_1, r_2 \in L_0$, $s, s_1, s_2 \in L_1$, we have the following properties:

(1) $C_{\eta,\eta^*}(1_\emptyset, r, s) = 1_\emptyset$.
(2) $f \leqslant C_{\eta,\eta^*}(f, r, s)$.
(3) *If* $f_1 \leqslant f_2$, *then* $C_{\eta,\eta^*}(f_1, r, s) \leqslant C_{\eta,\eta^*}(f_2, r, s)$.
(4) $C_{\eta,\eta^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \leqslant C_{\eta,\eta^*}(f_1, r, s) \oplus C_{\eta,\eta^*}(f_2, r_1, s_1)$.
(5) *If* $r_1 \leqslant r_2$ *and* $s_1 \geqslant s_2$, *then* $C_{\eta,\eta^*}(f, r_1, s_1) \leqslant C_{\eta,\eta^*}(f, r_2, s_2)$.
(6) *If* (X, η, η^*) *is double topogenous space, then*$C_{\eta,\eta^*}(C_{\eta,\eta^*}(f, r, s), r, s) \leqslant C_{\eta,\eta^*}(f, r, s)$.

Proof. (1) Since $\eta(1_\emptyset, 1_\emptyset) = \top$ and $\eta^*(1_\emptyset, 1_\emptyset) = \perp$ $C_{\eta,\eta^*}(1_\emptyset, r, s) = 1_\emptyset$.

(2) Since $\eta(g,f^*) \neq \perp$ and $\eta^*(g,f^*) \neq \top$, $g \leqslant f^*$ implies $f \leqslant C_{\eta,\eta^*}(f, r, s)$.

(3) and (5) are easily proved.

(4) Suppose that there exists $f_1, f_2 \in L^X\psi$ such that

$$C_{\eta,\eta^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1)\imath\, C_{\eta,\eta^*}(f_1, r, s) \oplus C_{\eta,\eta^*}(f_2, r_1, s_1).$$

By the definition of C_{η,η^*} there exists g_1, $g_2 \in L^X$ with $\eta(g_1, f_1^*)\imath r$, $\eta^*(g_1, f_1^*) \leqslant s$, $\eta(g_2, f_2^*)\imath r_1$ and $\eta^*(g_2, f_2^*) \leqslant s_1$ such that $C_{\eta,\eta^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1)\imath g_1^* \oplus g_2^*$.

On the other hand,

$$\begin{aligned}{}_\Psi\eta(g_1 \odot g_2, (f_1 \oplus f_2)^*) &\geqslant \eta(g_1 \odot g_2, f_1^*) \odot \eta(g_1 \odot g_2, f_2^*) \quad (\text{by LST6}) \\ &\geqslant \eta(g_1, f_1^*) \odot \eta(g_2, f_2^*) \qquad (\text{by LST4}) \\ &\eta r \odot r_1, \\ \eta^*(g_1 \odot g_2, (f_1 \oplus f_2)^*) &\leqslant \eta^*(g_1 \oplus g_2, f_1^*) \oplus \eta^*(g_1 \odot g_2, f_2^*) \quad (\text{by LST6}) \\ &\leqslant \eta^*(g_1, f_1^*) \oplus \eta^*(g_2, f_2^*) \qquad (\text{by LST4}) \\ &\leqslant s \oplus s_1.\end{aligned}$$

It implies $C_{\eta,\eta^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \leqslant (g_1 \odot g_2)^* = g_1^* \oplus g_2^*\Psi$. It is a contradiction.

(6) Let $\eta(g,f^*) \not< r$ and $\eta^*(g,f^*) \leqslant s$. Then $g^* \geqslant C_{\eta,\eta^*}(f, r, s)$. Since (X, η, η^*) is double fuzzy topogenous space, by (LS2) of Definition 2.7, there exists (η, η^*) such that $\eta \leqslant \eta \circ \eta$ and $\eta^* \geqslant \eta^* \circ \eta^*$. It follows

$$\eta(g,f^*) \leqslant \eta \circ \eta(g,f^*) \quad \text{and} \quad \eta^*(g,f^*) \geqslant \eta^* \circ \eta^*(g,f^*).$$

Since $\eta \circ \eta(g,f^*) \not< r$ and $\eta^* \circ \eta^*(g,f^*) \not< s$, there exists $h \in L^X$ such that

$$\eta\circ\eta(g,f^*) \geqslant \eta(g,h) \odot \eta(h,f^*) \not< r \quad \text{and}$$
$$\eta^* \circ \eta^*(g,f^*) \leqslant \eta^*(g,h) \oplus \eta^*(h,f^*) \leqslant s.$$

Hence, $g^* \geqslant C_{\eta,\eta^*}(h^*, r, s)$ and $h^* \geqslant C_{\eta,\eta^*}(f, r, s)$. Thus, $g^* \geqslant C_{\eta,\eta^*}(C_{\eta,\eta^*}(f, r, s), r, s)$. So,

$$C_{\eta,\eta^*}(C_{\eta,\eta^*}(f, r, s), r, s) \leqslant C_{\eta,\eta^*}(f, r, s). \qquad \square$$

Theorem 2.10. *Let (X, η, η^*) be a double fuzzy topogenous order space. Define the maps $\mathcal{T}_\eta, \mathcal{T}_{\eta^*}^* : L^X \to L$ by*

$$\begin{aligned}&\leftarrow\leftarrow \mathcal{T}_\eta(f) = \vee\{r \in L_0 \mid C\eta(f^*, r, s) \leqslant f^*\}, \\ &\mathcal{T}_{\eta^*}^*(f) = \wedge\{s \in L_1 \mid C\eta(f^*, r, s) \leqslant f^*\}.\end{aligned}$$

Then, $(\mathcal{T}_\eta, \mathcal{T}_{\eta^*}^*)$ is double fuzzy topology on X induced by (η, η^*).

Proof. (LO1) clear.

(LO2) $C_{\eta,\eta^*}(1_\emptyset, r, s) = 1_\emptyset$ and $C_{\eta,\eta^*}(1_X, r, s) = 1_X$ for all $r \in L_0, s \in L_1$, $\mathcal{T}_\eta(1_\emptyset) = \mathcal{T}_\eta(1_X) = \top$ and $\mathcal{T}_{\eta^*}^*(1_\emptyset) = \mathcal{T}_{\eta^*}^*(1_X) = \perp$.

(LO3) Suppose there exist $f_1, f_2 \in L^X$such that

$\mathcal{T}_\eta(f_1 \odot f_2) \not\geqslant \mathcal{T}_\eta(f_1) \odot \mathcal{T}_\eta(f_2)$ and $\mathcal{T}^*_{\eta^*}(f_1 \odot f_2) \not\leqslant \mathcal{T}^*_{\eta^*}(f_1) \oplus \mathcal{T}^*_{\eta^*}(f_2)$.

By the definition of $(\mathcal{T}_\eta, \mathcal{T}^*_{\eta^*})$ there exists $r_i \in L_0$, $s_i \in L_1$ with $f_i^* \geqslant C_{\eta,\eta^*}(f_i^*, r_i, s_i)$, $i = 1, 2$ such that

$\mathcal{T}_\eta(f_1 \odot f_2) \not\geqslant r_1 \odot r_2$ and $\mathcal{T}^*_{\eta^*}(f_1 \odot f_2) \not\leqslant s_1 \oplus s_2$.

Put $r = r_1 \odot r_2$ and $s = s_1 \oplus s_2$. By (4–5) of Theorem 2.9, we have

$C_{\eta,\eta^*}((f_1 \odot f_2)^*, r, s) \leqslant (f_1 \odot f_2)^*$.

Consequently, $\mathcal{T}_\eta(f_1 \odot f_2) \geqslant r$ and $\mathcal{T}^*_{\eta^*}(f_1 \odot f_2) \leqslant s$. Hence (LO3) holds.

(LO4) Suppose there exists a family $\{f_j \in L^X | j \in \Gamma\}$ such that

$\mathcal{T}_\eta(\bigvee_{i\in\Gamma} f_i) \not\geqslant \bigwedge_{i\in\Gamma} \mathcal{T}_\eta(f_i)$ or $\mathcal{T}^*_{\eta^*}(\bigvee_{i\in\Gamma} f_i) \not\leqslant \bigvee_{i\in\Gamma} \mathcal{T}^*_{\eta^*}(f_i)$.

For each $j \in \Gamma$, there exists $r_j \in L_0, s_j \in L_1$ *with* $f_j^* \geqslant C_{\eta,\eta^*}(f_j^*, r_j, s_j)$ such that

$\mathcal{T}_\eta(\bigvee_{j\in\Gamma} f_j) \not\geqslant \bigwedge_{j\in\Gamma} r_j$ or $\mathcal{T}^*_{\eta^*}(\bigvee_{j\in\Gamma} f_j) \not\leqslant \bigvee_{j\in\Gamma} s_j$.

Put $r = \bigwedge_{j\in\Gamma} r_j$ and $s = \bigvee_{j\in\Gamma} s_j$. By (4–5) of Theorem 2.9, we have $C_{\eta,\eta^*}((\bigvee_{j\in\Gamma} f_j)^*, r, s) \leqslant (\bigvee_{j\in\Gamma} f_j)^*$.

Consequently, $\mathcal{T}_\eta(\bigvee_{j\in\Gamma} f_j) \geqslant r$ and $\mathcal{T}^*_{\eta^*}(\bigvee_{j\in\Gamma} f_j) \leqslant s$. Hence (LO4) holds. □

Definition 2.11.

Let (X, η_1, η_1^*) and (Y, η_2, η_2^*) be double fuzzy topogenous order spaces. A function $\Psi\phi : (X, \eta_1, \eta_1^*) \to (Y, \eta_2, \eta_2^*)$ is said to be topogenous continuous if $\Psi\eta_2(f, g) \leqslant \eta_1(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g))$ and $\eta_2^*(f, g) \geqslant \eta_1^*(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g))$, Ψ *for each* $f, g \in L^Y$.

Theorem 2.12. Let (X, η_1, η_1^*), (Y, η_2, η_2^*) and (Z, η_3, η_3^*) be double fuzzy topogenous order spaces. If $\Phi : (X, \eta_1, \eta_1^*) \to (Y, \eta_2, \eta_2^*)$ and $\Psi : (Y, \eta_2, \eta_2^*) \to (Z, \eta_3, \eta_3^*)$ are topogenous continuous, then $\Psi \circ \Phi : (X, \eta_1, \eta_1^*) \to (Z, \eta_3, \eta_3^*)$ is topogenous continuous.

Proof.

It follows that, for each $f, g \in L^Z$

$$\begin{aligned}
\eta_1((\psi\circ\phi)_L^{\leftarrow}(f), (\psi\circ\phi)_L^{\leftarrow}(g)) &= \eta_1\big(\phi_L^{\leftarrow}(\psi_L^{\leftarrow}(f)), \phi_L^{\leftarrow}(\psi_L^{\leftarrow}(g))\big)\\
&\geqslant \eta_2\big(\psi_L^{\leftarrow}(f), \psi_L^{\leftarrow}(g)\big)\\
&\geqslant \eta_3(f, g),\\
\eta_1^*((\psi\circ\phi)_L^{\leftarrow}(f), (\psi\circ\phi)_L^{\leftarrow}(g)) &= \eta_1^*\big(\phi_L^{\leftarrow}(\psi_L^{\leftarrow}(f)), \phi_L^{\leftarrow}(\psi_L^{\leftarrow}(g))\big)\\
&\leqslant \eta_2^*(\psi_L^{\leftarrow}(f), \psi_L^{\leftarrow}(g))\\
&\leqslant \eta_3^*(f, g). \quad \square
\end{aligned}$$

Theorem 2.13. *Let* (X, η_1, η_1^*) *and* (Y, η_2, η_2^*) *be double fuzzy topogenous order spaces. If* $\Phi : (X, \eta_1, \eta_1^*) \to (Y, \eta_2, \eta_2^*)$ Ψ *is topogenous continuous, then it satisfies the following statements:*

(1) $\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(f, r, s)) \leqslant C_{\eta_2,\eta_2^*}(\phi_L^{\rightarrow}(f), r, s)$, *for each* $f \in L^X$, $r \in L_0$, $s \in L_1$,

(2) $C_{\eta_1,\eta_1^*}(\phi_L^{\leftarrow}(g), r, s) \leqslant \phi_L^{\rightarrow}(C_{\eta_2,\eta_2^*}(g, r, s))$ *for each* $g \in L^X$, $r \in L_0$, $s \in L_1$,

(3) $\phi : (X, \mathcal{T}_{\eta_1}, \mathcal{T}^*_{\eta_1^*}) \to (Y, \mathcal{T}_{\eta_2}, \mathcal{T}^*_{\eta_2^*})$ *is fuzzy continuous.*

Proof. (1) Suppose there exist $f \in L^X$, $r \in L_0$, $s \in L_1$ such that

$\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(f, r, s)) \cdot C_{\eta_2,\eta_2^*}(\phi_L^{\rightarrow}(f), r, s)$.

By the definition of C_{η_2,η_2^*} there exists $g \in L^Y$ with $\eta_2(g, (\phi_L^{\rightarrow}(f))^*)r$ and $\eta_2^*(g, (\phi_L^{\rightarrow}(f))^*) \leqslant s$ such that

$\Psi\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(f, r, s)) \not\leqslant g^*$. (A)

By the topogenous continuity of ϕ we have,

$\eta_1(\phi_L^{\leftarrow}(g), \phi_L^{\leftarrow}((\phi_L^{\rightarrow}(f))^*) \geqslant \eta_2(g, (\phi_L^{\rightarrow}(f))^*)\eta r$,

$\eta_1^*(\phi_L^{\leftarrow}(g), \phi_L^{\leftarrow}((\phi_L^{\rightarrow}(f))^*) \leqslant \eta_2^*(g, (\phi_L^{\rightarrow}(f))^*) \leqslant s$.

Since $\eta_1(\phi_L^{\leftarrow}(g), f^*) \geqslant \eta_1(\phi_L^{\leftarrow}g, \phi_L^{\leftarrow}((\phi_L^{\rightarrow}(f))^*)$ and $\eta_1^*(\phi_L^{\leftarrow}(g), f^*) \leqslant \eta_1^*(\phi_L^{\leftarrow}(g), \phi_L^{\leftarrow}((\phi_L^{\rightarrow}(f))^*)$ we have $C_{\eta_1,\eta_1^*}(f, r, s) = (\phi_L^{\leftarrow}(g))^* = \phi_L^{\leftarrow}(g^*)$. Thus $\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(f, r, s)) \leqslant g^*$. It is a contradiction for equation (A).

(2) For each $g \in L^Y$, $r \in L_0$ and $s \in L_1$, put $f = \phi_L^{\leftarrow}(g)$. From (1),

$$\begin{aligned}
\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(\phi_L^{\leftarrow}(g), r, s)) &\leqslant C_{\eta_2,\eta_2^*}(\phi_L^{\rightarrow}(\phi_L^{\leftarrow}(g)), r, s)\\
&\leqslant C_{\eta_2,\eta_2^*}(g, r, s).
\end{aligned}$$

It implies

$$\begin{aligned}
C_{\eta_1,\eta_1^*}(\phi_L^{\leftarrow}(g), r, s) &\leqslant \phi_L^{\leftarrow}(\phi_L^{\rightarrow}(C_{\eta_1,\eta_1^*}(\phi_L^{\leftarrow}(g), r, s)))\\
&\leqslant \phi_L^{\leftarrow}(C_{\eta_2,\eta_2^*}(g, r, s)).
\end{aligned}$$

(3) From (2), $C_{\eta_2,\eta_2^*}(g, r, s) = g$ implies $C_{\eta_1,\eta_1^*}(\phi_L^{\leftarrow}(g), r, s) = \phi_L^{\leftarrow}(g)$. It is easily proved from Theorem 2.10. □

3. Double fuzzy quasi-proximities

Definition 3.1 (Cetkin and Aygun, 2010). A maps $\delta, \delta^* : L^X \times L^X \to L$ is called a double fuzzy quasi-proximity on X if it satisfies the following axioms:

(LP1) $\delta(f, g) \leqslant (\delta^*(f, g))^*, \forall f, g \in L^X$.
(LP2) $\delta(1_X, 1_\emptyset) = \delta(1_\emptyset, 1_X) = \bot$ and $\delta^*(1_X, 1_\emptyset) = \delta^*(1_\emptyset, 1_X) = \top$. (LP3) If $\delta(f, g) \neq \top$ and $\delta^*(f, g) \neq \bot$ then $f \leqslant g^*$.
item(LO4) If $f \leqslant g$ then $\delta(f, h) \leqslant \delta(g, h)$ and $\delta^*(f, h) \geqslant \delta^*(g, h)$,
(LP5) $\delta(f_1 \odot f_2, g_1 \oplus g_2) \leqslant \delta(f_1, g_1) \oplus \delta(f_2, g_2)$ and $\delta^*(f_1 \odot f_2, g_1 \oplus g_2) \geqslant \delta^*(f_1, g_1) \odot \delta^*(f_2, g_2)$.
(LP6) For any $f, g \in L^X$, there exists $h \in L^X$such that $\delta(f, g) \geqslant \bigwedge_{h\in L^X} \{\delta(f, h) \oplus \delta(h^*, g)\}$ and $\delta^*(f, g) \leqslant \bigvee_{h\in L^X} \{\delta^*(f, h) \odot \delta^*(h^*, g)\}$. The triple $(X, \delta, \delta^*)$is double fuzzy quasi-proximity space.

A double fuzzy quasi-proximity space (X, δ, δ^*) is double fuzzy proximity space if it satisfies:

(LP) $\delta(f, g) = \delta(g, f)$ and $\delta^*(f, g) = \delta^*(g, f)$.

Proposition 3.2

(1) *Let* (X, η, η^*) *be a double fuzzy (resp. symmetric) topogenous space and let the maps* $\delta_\eta, \delta^*_{\eta^*} : L^X \times L^X \to L$ *defined by* $\delta_\eta(f, g) = (\eta(f, g^*))^*$ *and* $\delta^*_{\eta^*}(f, g) =$

$(\eta^*(f,g^*))^*$, $\forall f,\ g \in L^X$. *Then* $(\delta_\eta, \delta^*_{\eta^*})$ *is double fuzzy quasi-proximity space (resp. double fuzzy proximity space) on* X.

(2) *Let* (δ, δ^*) *be a double fuzzy quasi-proximity space(resp. double fuzzy proximity space) on and let the mappings* $\eta_\delta, \eta^*_{\delta^*} : L^X \times L^X \to L$ *defined by* $\eta_\delta(f,g) = (\delta(f,g^*))^*$ *and* $\eta^*_{\delta^*}(f,g) = (\delta^*(f,g^*))^*$, $\forall f,\ g \in L^X$. *Then* $(\eta_\delta, \eta^*_{\delta^*})$ *is double fuzzy (resp. symmetric) topogenous space.*

(3) $(\delta, \delta^*) = (\delta_{\eta_\delta}, \delta^*_{\eta^*_{\delta^*}})$ *and* $(\eta_{\delta_\eta}, \eta^*_{\delta^*_{\eta^*}}) = (\eta, \eta^*)$.

Proof. (1) Since $\eta \circ \eta \geqslant \eta$ and $\eta^* \circ \eta^* \leqslant \eta^*$.

$$\begin{aligned}
\delta_\eta(f,g) &= (\eta(f,g^*))^* \geqslant ((\eta\circ\eta)(f,g^*))^* \\
&\geqslant (\bigvee_{h\in L^X}\{(\eta(f,h)\odot\eta(h,g^*)\})^* \\
&= \bigwedge_{h\in L^X}\{(\eta(f,h))^* \oplus (\eta(h,g))^*\} \Psi \\
&= \bigwedge_{h\in L^X}\{\delta_\eta(f,h^*) \oplus \delta_\eta(h,g)\}, \\
\delta^*_{\eta^*}(f,g) &= (\eta^*(f,g^*))^* \leqslant ((\eta^*\circ\eta^*)(f,g^*))^* \\
&\leqslant (\bigwedge_{h\in L^X}\{(\eta^*(f,h) \oplus \eta^*(h,g^*)\})^* \\
&= \bigvee_{h\in L^X}\{(\eta^*(f,h))^* \odot (\eta(h,g))^*\} \\
&= \bigvee_{h\in L^X}\{\delta^*_{\eta^*}(f,h^*) \odot \delta^*_{\eta^*}(h,g)\}.
\end{aligned}$$

Let (X, δ, δ^*) be a double fuzzy symmetric topogenous space. Then

$$\begin{aligned}
&\delta_\eta(f,g) = (\eta(f,g^*))^* = (\eta(g,f^*)^* = \delta_\eta(f,g), \\
&\delta^*_{\eta^*}(f,g) = (\eta^*(f,g^*))^* = (\eta^*(g,f^*)^* = \delta^*_{\eta^*}(f,g). \Psi
\end{aligned}$$

Others are easily proved. □

(2) and (3) are easily proved.

From Theorem 2.9 and 10, we can obtain the following theorems.

Theorem 3.3. *Let* (δ, δ^*) *be a double fuzzy quasi-proximity space. The mapping* $\Psi C_{\delta,\delta^*} : L^X \times L_0 \times L_1 \to L^X$ *defined by*

$$\Psi C_{\delta,\delta^*}(f,r,s) = \wedge\{g^* \in L^X : \delta(g,f)r^* \quad \text{and } \delta(g,f) \geqslant s\}.$$

For each $f, f_1, f_2 \in L^X$, $r, r_1 \in L_0$ and $s, s_1 \in L_1$, we have the following properties:

(1) $C_{\delta,\delta^*}(1_\emptyset, r, s) = 1_\emptyset$.
(2) $f \leqslant C_{\delta,\delta^*}(f,r,s)$.
(3) If $f_1 \leqslant f_2$, *then* $C_{\delta,\delta^*}(f_1,r,s) \leqslant C_{\delta,\delta^*}(f_2,r,s)$.
(4) $C_{\delta,\delta^*}(f_1 \oplus f_2, r\odot r_1, s \oplus s_1) \leqslant C_{\delta,\delta^*}(f_1,r,s) \oplus C_{\delta,\delta^*}(f_2,r_1,s_1)$.
(5) *If* $r \leqslant r_1$ *and* $s \geqslant s_1$ *then* $C_{\delta,\delta^*}(f,r,s) \leqslant C_{\delta,\delta^*}(f,r_1,s_1)$.
(6) $C_{\delta,\delta^*}(C_{\delta,\delta^*}(f,r,s),r,s) \leqslant C_{\delta,\delta^*}(f,r,s)$.

Theorem 3.4. *Let* (X, δ, δ^*) *be a double fuzzy quasi-proximity space. Define the maps* $\mathcal{T}_\delta, \mathcal{T}^*_{\delta^*} : L^X \to L$ *by*

$$\begin{aligned}
&\mathcal{T}_\delta(f) = \vee\{r \in L_0 : C_{\delta,\delta^*}(f^*,r,s) \leqslant f^*\}, \\
&\mathcal{T}^*_{\delta^*}(f) = \wedge\{s \in L_1 : C_{\delta,\delta^*}(f^*,r,s) \leqslant f^*\}.
\end{aligned}$$

Then $(\mathcal{T}_\delta, \mathcal{T}^*_{\delta^*})$ *is double fuzzy topology on induced by* (δ, δ^*).

Definition 3.5. Let $(X, \delta_1, \delta_1^*)$ and $(Y, \delta_2, \delta_2^*)$ be double fuzzy quasi-proximity spaces. A map $\phi : (X, \delta_1, \delta_1^*) \to (Y, \delta_2, \delta_2^*)$ is said to be quasi-proximity continuous if

$$\begin{aligned}
\delta_2(f,g) \geqslant \delta_1(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g)) \quad &\text{or } \delta_2^*(f,g) \\
\leqslant \delta_1^*(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g)) &\forall f, \quad g \in L^Y.
\end{aligned}$$

Theorem 3.6. *Let* $(X, \delta_1, \delta_1^*)$ *and*$(Y, \delta_2, \delta_2^*)$ *be double fuzzy quasi-proximity spaces. A map* $\phi : (X, \delta_1, \delta_1^*) \to (Y, \delta_2, \delta_2^*)$ *is quasi-proximity continuous iff* $\phi : (X, \eta_{\delta_1}, \eta^*_{\delta_1^*}) \to (Y, \eta_{\delta_2}, \eta^*_{\delta_2^*})$ *is topogenous continuous.*

Proof. For all $f, g \in L^Y$

$$\begin{aligned}
&\delta_2(f,g) \geqslant \delta_1(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g)) \Longleftrightarrow (\eta_{\delta_2}(f,g^*))^* \geqslant (\eta_{\delta_1}(\phi_L^{\leftarrow}(f), (\phi_L^{\leftarrow}(g))^*)^* \\
&\qquad \Longleftrightarrow \eta_{\delta_2}(f,g^*) \leqslant \eta_{\delta_1}(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g^*)), \\
&\delta_2^*(f,g) \leqslant \delta_1^*(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g)) \Longleftrightarrow (\eta^*_{\delta_2^*}(f,g^*))^* \leqslant (\eta^*_{\delta_1^*}(\phi_L^{\leftarrow}(f), (\phi_L^{\leftarrow}(g))^*)^* \\
&\qquad \Longleftrightarrow \eta^*_{\delta_2^*}(f,g^*) \geqslant \eta^*_{\delta_1^*}(\phi_L^{\leftarrow}(f), \phi_L^{\leftarrow}(g^*)). \quad \square
\end{aligned}$$

4. Double fuzzy quasi-uniform spaces and double fuzzy syntopogenous spaces

Now we recall some notions and terminologies about double fuzzy quasi-uniform spaces used in this paper.

Let $\Omega(L^X)$ denote the family of all mappings $a : L^X \to L^X$ with the following properties:

(1) $f \leqslant a(f)$ *for each* $f \in L^X$,
(2) $a(\bigvee_{i\in\Gamma} f_i) = \bigvee_{i\in\Gamma} a(f_i)$, *for each* $f_i \in L^X$.

For $a, b \in \Omega(L^X)$, we have that a^{-1}, $a \odot$ b and $a \circ b \in \Omega(L^X)$ by

$$\begin{aligned}
&a^{-1}(f) = \wedge\{g \mid a(g^*) \leqslant f^*\}, \\
&(a\odot b)(f) = \wedge\{a(f_1) \oplus b(f_2) \mid f_1 \oplus f_2 = f\} \text{ and } (a\circ b)(f) = a(b(f)).
\end{aligned}$$

Definition 4.1. The mappings $\mathcal{U}, \mathcal{U}^* : \Omega(L^X) \to L$ is called a double fuzzy quasi-uniformity on X if it satisfies the following conditions: for $a, b \in \Omega(L^X)$,

(LU1) $\mathcal{U}(a) \leqslant (\mathcal{U}^*(a))^*$, for all $a \in \Omega(L^X)$,
(LU2) $\mathcal{U}(a \odot b) \geqslant \mathcal{U}(a) \odot \mathcal{U}(b)$ and $\mathcal{U}^*(a \odot b) \leqslant \mathcal{U}^*(a) \oplus \mathcal{U}^*(b)$.
(LU3) *there exists* $a \in \Omega(L^X)$ such that $\mathcal{U}(a) = \top$ and $\mathcal{U}^*(a) = \perp$
(LU4) $\mathcal{U}(a) \leqslant \vee\{\mathcal{U}(b) : b \circ b \leqslant a\}$ and $\mathcal{U}^*(b) \geqslant \{\mathcal{U}^*(b) : b \circ b \leqslant a\}$.

The triple $(X, \mathcal{U}, \mathcal{U}^*)$ is said to be double fuzzy quasi-uniform space.

A double fuzzy quasi-uniform space $(X, \mathcal{U}, \mathcal{U}^*)$ is said to be a double fuzzy uniform space if it satisfies.

(LU) $\mathcal{U}(a) = \mathcal{U}(a^{-1})$ and $\mathcal{U}^*(a) = \mathcal{U}^*(a^{-1})$.

Definition 4.2. The mappings $\mathcal{B}, \mathcal{B}^* : \Omega(L^X) \to L$ is called a double fuzzy quasi-uniform base on X if it satisfies the following conditions: for $a, b \in \Omega(L^X)$,

(LUB1) $\mathcal{B}(a) \leqslant (\mathcal{B}^*(a))^*$, for all $a \in \Omega(L^X)$,
(LUB2) $\mathcal{B}(a) \odot \mathcal{B}(b) \geqslant \vee\{\mathcal{B}(b) : b \leqslant a \odot b\}$ and $\mathcal{B}^*(a) \oplus \mathcal{B}^*(b) \leqslant \wedge\{\mathcal{B}^*(b) : b \leqslant a \odot b\}$.
(LUB3) there exists $a \in \Omega(L^X)$ such that $\mathcal{B}(a) = \top$ and $\mathcal{B}^*(a) = \bot$
(LUB4) $\mathcal{B}(a) \leqslant \vee\{\mathcal{B}(b) : b \circ b \leqslant a\}$ and $\mathcal{B}^*(a) \geqslant \wedge \{\mathcal{B}^*(b) : b \circ b \leqslant a\}$.

A double fuzzy quasi-uniform base $(\mathcal{B}, \mathcal{B}^*)$ on X is said to be double fuzzy uniform base if it satisfies

(LUB) $\mathcal{B}(a) \leqslant \vee\{\mathcal{B}(b) : b \leqslant a^{-1}\}$ and $\mathcal{B}^*(a) \geqslant \wedge\{\mathcal{B}^*(b) : b \leqslant a^{-1}\}$.

Theorem 4.3. *Let $\mathcal{B}, \mathcal{B}^* : \Omega(L^X) \to L$ be a double fuzzy uniform base on X.*

Define $\mathcal{U}_{\mathcal{B}}, \mathcal{U}^*_{\mathcal{B}^*} : \Omega(L^X) \to L$ as

$\mathcal{U}_{\mathcal{B}}(a) = \vee\{\mathcal{B}(b) : b \leqslant a\}$ and $\mathcal{U}^*_{\mathcal{B}^*}(a) = \wedge\{\mathcal{B}^*(b) : b \leqslant a\}$.

Then $(\mathcal{U}_{\mathcal{B}}, \mathcal{U}^*_{\mathcal{B}^*})$ is double fuzzy uniformity on it X.

Proof. (LU) For each $\Psi a \in \Omega(L^X)$

$$\mathcal{U}_{\mathcal{B}}(a) = \vee\{\mathcal{B}(b) : b \leqslant a\} \leqslant \bigvee_{b \leqslant a}\{\bigvee_{c \leqslant b^{-1}} \mathcal{B}(c)\} \qquad (byLUB)$$
$$\leqslant \bigvee_{b \leqslant a} \mathcal{U}_{\mathcal{B}}(b^{-1}) = \bigvee_{b^{-1} \leqslant a^{-1}} \mathcal{U}_{\mathcal{B}}(b^{-1}) \leqslant \mathcal{U}_{\mathcal{B}}(a^{-1}),$$
$$\mathcal{U}^*_{\mathcal{B}^*}(a) = \wedge\{\mathcal{B}^*(b) : b \leqslant a\} \geqslant \bigwedge_{b \leqslant a}\{\bigwedge_{c \leqslant b^{-1}} \mathcal{B}(c)\} \qquad (byLUB)$$
$$\geqslant \bigwedge_{b \leqslant a} \mathcal{U}^*_{\mathcal{B}^*}(b^{-1}) = \bigwedge_{b^{-1} \leqslant a^{-1}} \mathcal{U}^*_{\mathcal{B}^*}(b^{-1}) \geqslant \mathcal{U}^*_{\mathcal{B}^*}(a^{-1}).$$

Since $a = (a^{-1})^{-1}$, we have $\mathcal{U}_{\mathcal{B}}(a^{-1}) \leqslant \mathcal{U}_{\mathcal{B}}(a)$ and $\mathcal{U}^*_{\mathcal{B}^*}(a^{-1}) \geqslant \mathcal{U}^*_{\mathcal{B}^*}(a)$. □

Other cases are easily proved.

Definition 4.4. Let Υ_X be a double fuzzy biperfect syntopogenous structure on X. The mappings $\mathcal{S}, \mathcal{S}^* : \Upsilon_X \to L$ is called double fuzzy syntopogenous structure on X if it satisfies the following conditions: for for (η, η^*), (η_1, η_1^*), $(\eta_2, \eta_2^*) \in \Upsilon_X$.

(LT1) $\mathcal{S}(\eta, \eta^*) \leqslant (\mathcal{S}^*(\eta, \eta^*))^*$.
(LT2) There exists $(\eta, \eta^*) \in \Upsilon$ such that $\mathcal{S}(\eta, \eta^*) = \top$ and $\mathcal{S}^*(\eta, \eta^*) = \bot$
(LT3) $\mathcal{S}(\eta_1, \eta_1^*) \odot \mathcal{S}(\eta_2, \eta_2^*) \leqslant \vee\{\mathcal{S}(\eta, \eta^*) : \eta_1, \eta_2 \leqslant \eta$ and $\eta_1^*, \eta_2^* \geqslant \eta^*\}$ and $\mathcal{S}^*(\eta_1, \eta_1^*) \oplus \mathcal{S}^*(\eta_2, \eta_2^*) \geqslant \wedge\{\mathcal{S}^*(\eta, \eta^*) : \eta_1, \eta_2 \leqslant \eta$ and $\eta_1^*, \eta_2^* \geqslant \eta^*\}$.
(LT4) $\mathcal{S}(\eta, \eta^*) \leqslant \vee\{\mathcal{S}(\eta_1, \eta_1^*) : \eta_1 \circ \eta_2 \geqslant \eta$ and $\eta_1^* \circ \eta_2^* \leqslant \eta^*\}$ and $\mathcal{S}^*(\eta_1, \eta_1^*) \geqslant \wedge\{\mathcal{S}^*(\eta_1, \eta_1^*) : \eta_1 \circ \eta_2 \geqslant \eta$ and $\eta_1^*, \eta_2^* \leqslant \eta^*\}$.

The triple $(X, \mathcal{S}, \mathcal{S}^*)$ is said to be double fuzzy syntopogenous space.

A double fuzzy syntopogenous space $(X, \mathcal{S}, \mathcal{S}^*)$ is said to be double fuzzy symmetric syntopogenous space if it satisfies

(LST), $\mathcal{S}(\eta, \eta^*) \leqslant \vee\{\mathcal{S}(\xi, \xi^*) : \xi \geqslant \eta^s$ and $\xi^* \leqslant \eta^{*s}\}$,
$\mathcal{S}^*(\eta, \eta^*) \geqslant \wedge\{\mathcal{S}^*(\xi, \xi^*) : \xi \geqslant \eta^s$ and $\xi^* \leqslant \eta^{*s}\}$.

Lemma 4.5. *To every $a \in \Omega(L^X)$, we define $\eta_a, \eta_a^* : L^X \times L^X \to L$ as*

$$\eta_a(g,f) = \begin{cases} \top, & \text{if } f \geqslant a(g), \\ \bot & \text{otherwise,} \end{cases} \quad \eta_a^*(g,f) = \begin{cases} \bot, & \text{if } f \geqslant a(g), \\ \top & \text{otherwise.} \end{cases}$$

Then it satisfies the following properties:

(1) *The maps η_a, $\eta_a^* \in \Upsilon_X$ is double fuzzy biperfect topogenous order.*
(2) *If $a \leqslant b$, then $\eta_b \leqslant \eta_a$ and $\eta_b^* \leqslant \eta_a^*$.*
(3) *If $b \leqslant a_1 \odot a_2$, then η_{a_1}, $\eta_{a_2} \leqslant \eta_b$ and $\eta_{a_1}^*$, $\eta_{a_2}^* \geqslant \eta_b^*$.*
(4) *For each $a \in \Omega(L^X)$, we have $(\eta_a^s, \eta_a^{*s}) = (\eta_{a^{-1}}, \eta_{a^{-1}}^*)$.*
(5) *If $b \circ b \leqslant a$, then $\eta_b \circ \eta_b \geqslant \eta_a$ and $\eta_b^* \circ \eta_b^* \leqslant \eta_a^*$.*

Proof. (1) Since $\Psi a(1_X) = 1_X$ and $a(1_\emptyset) = 1_\emptyset$, then $\eta_a(1_X, 1_X) = \eta_a(1_\emptyset, 1_\emptyset) = \top\Psi$ and $\eta_a^*(1_X, 1_X) = \eta_a^*(1_\emptyset, 1_\emptyset) = \bot$ Let $\eta_a(g,f) \neq \bot$ and $\eta_a^*(g,f) \neq \top$. Then $\eta_a(g,f) = \top$ and $\eta_a^*(g,f) = \bot$ implies $g \leqslant a(g) \leqslant f$. Since $g \leqslant g_1$ and $f_1 \leqslant f$ implies $a(g) \leqslant a(g_1)$, then$\eta_a(g_1,f_1) \leqslant \eta_a(g,f)$ and $\eta_a^*(g_1,f_1) \geqslant \eta_a^*(g,f)$. To prove the biperfect condition, since $a(\bigvee_{i\in\Gamma} g_i) = \vee_{i\in\Gamma} a(g_i) \leqslant f$ iff $g_i \leqslant f$ for all $i \in \Gamma, \eta_a(\bigvee_{i\in\Gamma} g_i, f) \geqslant \bigwedge_{i\in\Gamma} \eta_a(g_i,f)$ and $\eta_a^*(\bigvee_{i\in\Gamma} g_i, f) \leqslant \bigvee_{i\in\Gamma} \eta_a^*(g_i,f)$. Since $g \leqslant \bigwedge_{i\in\Gamma} f_j$ iff $g \leqslant f_j$ for all $j \in \Lambda$, $\eta_a(g, \bigwedge_{j\in\Lambda} f_j) \geqslant \bigwedge_{j\in\Lambda} \eta_a(g,f_j)$ and $\eta_a^*(g, \bigwedge_{j\in\Lambda} f_j) \leqslant \bigvee_{j\in\Lambda} \eta_a^*(g,f_j)$.

Others are similarly proved.

(2) Since $\Psi a(g) \leqslant b(g)$, $\eta_b \leqslant \eta_a$ and $\eta_b^* \geqslant \eta_a^*$.

(3) Since $a_1 \odot a_2(g) \leqslant a_1(g) \oplus a_2(g)\Psi$ we have $a_1 \odot a_2 \leqslant a_1$. From (2), $\eta_{a_1} \leqslant \eta_b$ and $\eta_{a_1}^* \geqslant \eta_b^*$. Similary $\eta_{a_2} \leqslant \eta_b$ and $\eta_{a_2}^* \geqslant \eta_b^*$.$\Psi$

(4) It easily proved from $a^{-1}(g) \leqslant f$ iff $a(f^*) \leqslant g^*$.

(5) From (2), we only show that $\eta_b \odot \eta_b = \eta_{b\circ b}$ and $\eta_b^* \odot \eta_b^* = \eta_{b\circ b}^*$. Since $\eta_b \circ \eta_b(g,f) = \vee\{\eta_b(g,h) \odot \eta_b(h,f) : h \in L^X\}and\eta_b^* \circ \eta_b^*(g,f) = \wedge\{\eta_b^*(g,h) \oplus \eta_b^*(h,f) : h \in L^X\}$, we have

$$\eta_b \circ \eta_b(g,f) = \begin{cases} \top \text{ if } f \geqslant b(b(g)), \\ \bot \text{ otherwise,} \end{cases} \quad \eta_b^* \circ \eta_b^*(g,f) = \begin{cases} \bot \text{ if } f \geqslant b(b(g)), \\ \top \text{otherwise.} \end{cases}$$

From Lemma 4.5, we easily prove the following theorem. □

Theorem 4.6. *Let $\mathcal{B}, \mathcal{B}^* : \Omega(L^X) \to L$ be double fuzzy quasi-uniform (resp. double fuzzy uniform) base on X. Define $\mathcal{S}_{\mathcal{B}}, \mathcal{S}^*_{\mathcal{B}^*} : \Upsilon_X \to L$ as*

$\mathcal{S}_{\mathcal{B}}(\eta_a, \eta_{a^*}^*) = \mathcal{B}(a)$ and $\mathcal{S}^*_{\mathcal{B}^*}(\eta_a, \eta_{a^*}^*) = \mathcal{B}^*(a)$.

Then $(\mathcal{S}_{\mathcal{B}}, \mathcal{S}^*_{\mathcal{B}^*})$ is double fuzzy (resp. double fuzzy symmetric) syntopogenous structure on X.

Theorem 4.7. *Let $\mathcal{U}, \mathcal{U}^* : \Omega(L^X) \to L$ be a double fuzzy quasi-uniformity on X.*

The mapping $\mathcal{C}_{\mathcal{U},\mathcal{U}^*} : L^X \times L_0 \times L_1 \to L^X$, is defined by

$\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s) = \wedge\{a(f) : \mathcal{U}(a) \geqslant r \text{ and } \mathcal{U}^*(a) \leqslant s\}$.

For each $f, f_1, f_2 \in L^X$, $r, r_1, r_2 \in L_0$ and $s, s_1, s_2 \in L_1$, we have the following properties:

(1) $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(1_\emptyset, r, s) = 1_\emptyset$,
(2) $f \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)$,
(3) if $f_1 \leqslant f_2$, then $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1,r,s) \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_2,r,s)$,
(4) $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1,r,s) \oplus \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_2,r_1,s_1)$,
(5) if $r_1 \leqslant r_2$ and $s_1 \geqslant s_2$, then $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1,r_1,s_1) \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_2,r_2,s_2)$,
(6) $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s) \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)$.

Proof

(1) Since $\Psi a(1_\emptyset) = 1_\emptyset, \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(1_\emptyset,r,s) = 1_\emptyset$.
(2) Since $\Psi f \leqslant a(f)$, Ψ implies $\Psi f \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)$,
(3) and (5) are easily proved.
(4) Conversely, suppose there exist $f, f_1, f_2 \in L^X$, $r, r_1, r_2 \in L_0$ and $s, s_1, s_2 \in L_1$, such that

$\leftarrow \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \not\leqslant \rightarrow \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1,r,s) \oplus \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_2,r_1,s_1)$

There exist $a_1, a_2 \in \Omega(L^X)$ with $\mathcal{U}(a_1) \geqslant r$, $\mathcal{U}^*(a_1) \leqslant s$, $\mathcal{U}(a_2) \geqslant r_1$, $\mathcal{U}^*(a_2) \leqslant s_1$, such that $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1)\,\overline{a}(f_1) \oplus a(f_2)$.

On the other hand, $\mathcal{U}(a_1 \odot a_2) \geqslant \mathcal{U}(a_1) \odot \mathcal{U}(a_2) \geqslant r \odot r_1, \mathcal{U}^* \quad (a_1 \odot a_2) \leqslant \mathcal{U}^*(a_1) \oplus \mathcal{U}^*(a_2) \leqslant s \oplus s_1$ and $(a_1 \odot a_2)(f_1 \oplus f_2) \leqslant a_1(f_1) \oplus a_2(f_2)$, we have $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \leqslant (a_1 \odot a_2)(f_1 \oplus f_2) \leqslant a_1(f_1) \oplus a_2(f_2)$. It is a contradiction.

(6) Suppose there exist $\Psi f \in L^X$, $r \in L_0$ and $s \in L_1$ such that $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s) \not\leqslant \leftarrow \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)$.

There exists $a \in \Omega(L^X)\Psi$ with $\mathcal{U}(a) \geqslant r$ and $\mathcal{U}^*(a) \leqslant s$, $\leftarrow$ such that $\leftarrow \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant a(f)$. On the other hand, $\mathcal{U}(a) \geqslant r$ and $\mathcal{U}^*(a) \leqslant s$, by (LSU3), there exists $a_1 \in \Omega(L^X)$ $\leftarrow$ such that $a_1 \circ a_2 \leqslant a, \mathcal{U}(a_1) \geqslant r$ and $\mathcal{U}^*(a_1) \leqslant s$. Ψ Since $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s) \leqslant a_1(f)$, Ψ we have

$\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s) \leqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(a_1(f),r,s) \leqslant a_1(a_1(f)) \leqslant a(f)$.

Thus $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s),r,s) \leqslant a(f)$. It is a contradiction. □

Definition 4.8. Let $(X,\mathcal{U},\mathcal{U}^*)$ and $(Y,\mathcal{V},\mathcal{V}^*)$ be double fuzzy uniform (resp. double fuzzy quasi-uniform) spaces. A mapping $\Psi\Psi\ \phi : (X,\mathcal{U},\mathcal{U}^*) \to (Y,\mathcal{V},\mathcal{V}^*)$ is said to be uniformly continuous (resp. quasi-uniformly continuous) if

$\mathcal{V}(a) \leqslant \mathcal{U}(\phi_L^{\leftarrow}(a))$ and $\mathcal{V}^*(a) \geqslant \mathcal{U}^*(\phi_L^{\leftarrow}(a)), \quad \forall\ a \in \Omega_X$,

where $\Phi\ \phi_L^{\leftarrow}(a)(f) = \phi_L^{\leftarrow}(a(\phi_L^{\rightarrow}(f)))$ for all $f \in L^X$.

From Theorem 4.3, we easily prove the following theorem.

Definition 4.9. Let $(X,\mathcal{B}_1,\mathcal{B}_1^*)$ and $(Y,\mathcal{B}_1,\mathcal{B}_1^*)$ be double fuzzy quasi-uniform bases. If $\mathcal{B}_2(a) \leqslant \mathcal{B}_1(\phi_L^{\leftarrow}(a))$ and $\mathcal{B}_2^*(a) \geqslant \mathcal{B}_1^*(\phi_L^{\leftarrow}(a))$ for all $a \in \Omega(L^Y)$ then $\phi : (X,\mathcal{U}_{\mathcal{B}_1}, \mathcal{U}^*_{\mathcal{B}_1^*}) \to (Y,\mathcal{U}_{\mathcal{B}_2},\mathcal{U}^*_{\mathcal{B}_2^*})$ is quasi-uniformly continuous.

Theorem 4.10. *Let $(X,\mathcal{U},\mathcal{U}^*), (Y,\mathcal{V},\mathcal{V}^*)$ and $(Z,\mathcal{W},\mathcal{W}^*)$ be double fuzzy quasi-uniform spaces. If $\phi : (X,\mathcal{U},\mathcal{U}^*) \to (Y,\mathcal{V},\mathcal{V}^*)$ and $\psi : (Y,\mathcal{V},\mathcal{V}^*) \to (Z,\mathcal{W},\mathcal{W}^*)$ are quasi-uniformly continuous, then $\psi \circ \phi : (X,\mathcal{U},\mathcal{U}^*) \to (Z,\mathcal{W},\mathcal{W}^*)$ is quasi-uniformly continuous.*

Theorem 4.11. *Let $(X,\mathcal{U},\mathcal{U}^*)and(Y,\mathcal{V},\mathcal{V}^*)$ be double fuzzy quasi-uniform spaces. Let $\phi : (X,\mathcal{U},\mathcal{U}^*) \to (Y,\mathcal{V},\mathcal{V}^*)$ be quasi-uniformly continuous. Then:*

(1) $\phi_L^{\rightarrow}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)) \leqslant \mathcal{C}_{\mathcal{V},\mathcal{V}^*}(\phi_L^{\rightarrow}(f),r,s)$, *for each $f \in L^X$, $r \in L_0$, $s \in L_1$.*
(2) $\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(\phi_L^{\leftarrow}(g),r,s) \leqslant \phi_L^{\leftarrow}(\mathcal{C}_{\mathcal{V},\mathcal{V}^*}(g,r,s))$, *for each $g \in L^y$, $r \in L_0$, $s \in L_1$.*
(3) $\phi : (X,\mathcal{T}_{\mathcal{U}},\mathcal{T}^*_{\mathcal{U}^*}) \to (Y,\mathcal{T}_{\mathcal{V}},\mathcal{T}^*_{\mathcal{V}^*})$ *is fuzzy continuous.*

Proof. (1) Suppose there exist $f \in L^X$, $r \in L_0$ and $s \in L_1$ such that

$\phi_L^{\rightarrow}(\mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r,s)) \not\leqslant \mathcal{C}_{\mathcal{V},\mathcal{V}^*}(\phi_L^{\rightarrow}(f),r,s)$.

There exists $a \in \Omega(L^Y)$ with $\mathcal{V}(a) \geqslant r$ and $\mathcal{V}^*(a) \leqslant s$ such that $\mathcal{C}_{\mathcal{V},\mathcal{V}^*}(\phi_L^{\rightarrow}(f),r,s) \not\leqslant \overline{a}(\phi_L^{\rightarrow}(f))$.

On the other hand, ϕ is quasi-uniformly continuous,

$\mathcal{U}(\phi_L^{\leftarrow}(a)) \geqslant \mathcal{V}(a) \geqslant r \quad$ and $\mathcal{U}^*(\phi_L^{\leftarrow}(a)) \leqslant \mathcal{V}^*(a) \leqslant s$.

It implies $a(\phi_L^{\rightarrow}(f))(\phi(x)) = \phi_L^{\leftarrow}(a)(f)(x) \geqslant \mathcal{C}_{\mathcal{U},\mathcal{U}^*}(f,r)(x)$. It is a contradiction.

(2) and (3) are similarly proved as Theorem 2.13. □

Theorem 4.12. *Let $(X,\mathcal{S},\mathcal{S}^*)$ be a double fuzzy syntopogenous space. The mapping $\mathcal{C}_{\mathcal{S},\mathcal{S}^*} : L^X \times L_0 \times L_1 \to L^X$, is defined by*

$$\mathscr{C}_{\mathscr{S},\mathscr{S}^*}(f,r,s) = \wedge\{g : \eta(f,g) > \perp, \eta^*(f,g) \leqslant \top, \mathscr{S}(\eta,\eta^*) \geqslant r \text{ and } \mathscr{S}^*(\eta,\eta^*) \leqslant s\}.$$

For each $\Psi f, f_1, f_2 \in L^X$, $r, r_1, r_2 \in L_0$ and $s, s_1, s_2 \in L_1$,we have the following properties:

(1) $\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(1_\emptyset,r,s) = 1_\emptyset$,
(2) $f \leqslant \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r,s)$,
(3) if $f_1 \leqslant f_2, \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_1,r,s) \leqslant \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_2,r,s)$,
(4) $\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \leqslant \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_1,r,s) \oplus \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_2,r_1,s_1)$,
(5) if $r_1 \leqslant r_2$ and $s_1 \geqslant s_2$, then $\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r_1,s_1) \leqslant \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r_2,s_2)$,
(6) $\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r,s),r,s) = \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r,s)$.

Proof. (1) Since $\Psi\ \eta(1_\emptyset,1_\emptyset) = \top$ and $\eta^*(1_\emptyset,1_\emptyset) = \perp$ for $\mathcal{S}(\eta,\eta^*) = \top, \mathcal{S}^*(\eta,\eta^*) = \perp$,

$\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(1_\emptyset,r,s) = 1_\emptyset$,

(2) Since $\Psi f \leqslant g$ for $\eta(f,g) \perp$ and $\eta^*(f,g) \leqslant \top$, $f \leqslant \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f,r,s)$.

(3) and (5) are easily proved.

(4) Suppose there exist $f_1, f_2, \in L^X$, $r, r_1 \in L_0$ and $s, s_1 \in L_1$ such that

$$\mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \not\leqslant \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_1,r,s) \oplus \mathcal{C}_{\mathcal{S}},\mathcal{S}^*(f_2,r_1,s_1).$$

There exist,

$(\eta_1,\eta_1^*),(\eta_2,\eta_2^*) \in \Upsilon_X$ with $\mathcal{S}(\eta_1,\eta_1^*) \geqslant r, \mathcal{S}(\eta_2,\eta_2^*) \geqslant r_1, \mathcal{S}^*(\eta_1,\eta_1^*) \leqslant s$, $\mathcal{S}^*(\eta_2,\eta_2^*) \leqslant s_1, \eta_i(f_i,g_i) \not\leqslant \perp$ and $\eta_1^*(f_i,g_i) \leqslant \top$ such that, $\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \not\leqslant g_1 \oplus g_2$.

On the other hand, $\mathcal{S}(\eta_1,\eta_1^*) \odot \mathcal{S}(\eta_2,\eta_2^*) \geqslant r$ and $\mathcal{S}^*(\eta_1,\eta_1^*) \odot \mathcal{S}^*(\eta_2,\eta_2^*) \leqslant s$, by (LT2) of Definition 4.4, there exist $\Psi\eta \geqslant \eta_i, \eta^* \leqslant \eta_i^*, \mathcal{S}(\eta,\eta^*) \geqslant r$ and $\mathcal{S}^*(\eta,\eta^*) \leqslant s$ such that

$$\begin{aligned}\eta(f_1 \oplus f_2, g_1 \oplus g_2) &\geqslant \eta(f_1,g_1 \oplus g_2) \odot \eta(f_2,g_1 \oplus g_2)\\ &\geqslant \eta(f_1,g_1) \odot \eta(f_2,g_2)\\ &\geqslant \eta_1(f_1,g_1) \odot \eta_2(f_2,g_2) \not\leqslant \perp.\end{aligned}$$

Hence $C_{\mathcal{S},\mathcal{S}^*}(f_1 \oplus f_2, r \odot r_1, s \oplus s_1) \leqslant g_1 \oplus g_2$. It is a contradiction.

$$\begin{aligned}\eta^*(f_1 \oplus f_2, g_1 \oplus g_2) &\leqslant \eta^*(f_1,g_1 \oplus g_2) \odot \eta^*(f_2,g_1 \oplus g_2)\\ &\leqslant \eta^*(f_1,g_1) \odot \eta^*(f_2,g_2)\\ &\leqslant \eta_1^*(f_1,g_1) \oplus \eta_2^*(f_2,g_2) \leqslant \top.\end{aligned}$$

(6) Suppose there exist $\Psi f \in L^X$, $r \in L_0$ and $s \in L_1$ such that

$$\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f,r,s),r,s) \not\leqslant \mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f,r,s).$$

There exists $\Psi g \in L^X$ with $\mathcal{S}(\eta,\eta^*) \geqslant r$, $\mathcal{S}^*(\eta,\eta^*) \leqslant s$, $\eta(f,g) \not\leqslant \perp$ and $\eta^*(f,g) \leqslant \top$,

such that $\leftarrow \mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f,r,s) \leqslant g$. On the other hand, $\mathcal{S}(\eta,\eta^*) \not\leqslant r$ and $\mathcal{S}^*(\eta,\eta^*) \leqslant s$, $\leftarrow$ by (LT3) of Definition 4.4, there exists $(\zeta,\zeta^*) \in {}_X$ such that

$$(\zeta,\zeta^*) \circ (\zeta,\zeta^*) \geqslant (\eta,\eta^*), \quad \mathcal{S}(\zeta,\zeta^*) \geqslant r \quad \text{and } \mathcal{S}^*(\zeta,\zeta^*) \leqslant s.$$

Since $\zeta \circ \zeta(f,g) \not< \perp$ and $\zeta^* \circ \zeta^*(f,g) \leqslant \top$, here exists $\rho \in L^X$ such that $\zeta(f,\rho) \odot \zeta(\rho,g) \not< \perp$ and $\zeta^*(f,\rho) \oplus \zeta^*(\rho,g) \leqslant \top$. It implies $\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f,r,s) \leqslant \rho, \mathcal{C}_{\mathcal{S},\mathcal{S}^*}(\rho,r,s) \leqslant g$. Hence $\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f,r,s),r,s) \leqslant g$. Thus, $\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(\mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f,r,s),r,s) \leqslant g$. It is a contradiction. □

Definition 4.13. Let $(X,\mathcal{S},\mathcal{S}^*)$ be a double fuzzy syntopogenous space. Define the maps $\mathcal{T}_{\mathcal{S}},\mathcal{T}^*_{\mathcal{S}} : L^X \to L$ *by*

$$\mathcal{T}_{\mathcal{S}}(f) = \vee\{r \in L_0 : \mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f^*,r,s) \leqslant f^*\},$$
$$\mathcal{T}^*_{\mathcal{S}^*}(f) = \wedge\{s \in L_1 : \mathcal{C}_{\mathcal{S},\mathcal{S}^*}(f^*,r,s) \leqslant f^*\}.$$

Then $(\mathcal{T}_{\mathcal{S}},\mathcal{T}^*_{\mathcal{S}^*})$ is a double fuzzy topology on Xinduced by $(\mathcal{S},\mathcal{S}^*)$.

Theorem 4.14. *Let $(X,\mathcal{S}_1,\mathcal{S}_1^*)$ and $(Y,\mathcal{S}_2^*,\mathcal{S}_2^*)$ be double fuzzy syntopogenous space. A map $\phi : (X,\mathcal{S}_1,\mathcal{S}_1^*) \to (Y,\mathcal{S}_2^*,\mathcal{S}_2^*)$ is said to be syntopogenous continuous if for each $(\zeta,\zeta^*) \in {}_Y$, here exists $(\zeta,\zeta^*) \in {}_X$with $\eta(\phi_L^{\leftarrow}(g),\phi_L^{\leftarrow}(f)) \geqslant \zeta(g,f)$ and $\eta^*(\phi_L^{\leftarrow}(g),\phi_L^{\leftarrow}(f)) \leqslant \zeta^*(g,f)$such that $\mathcal{S}_2(\zeta,\zeta^*) \leqslant \mathcal{S}_1(\zeta,\zeta^*)$ and $\mathcal{S}_2^*(\zeta,\zeta^*) \geqslant \mathcal{S}_1^*(\zeta,\zeta^*)$.*

Theorem 4.15. *Let $(X,\mathcal{S}_1,\mathcal{S}_1^*)and(Y,\mathcal{S}_2^*,\mathcal{S}_2^*)$ be double fuzzy syntopogenous space. Let $\phi : (X,\mathcal{S}_1,\mathcal{S}_1^*) \to (Y,\mathcal{S}_2^*,\mathcal{S}_2^*)$ be syntopogenous continuous. Then we have the following properties:*

(1) $\phi_L^{\rightarrow}(\mathcal{C}_{\mathcal{S}_1,\mathcal{S}_1^*}(f,r,s) \leqslant \mathcal{C}_{\mathcal{S}_2,\mathcal{S}_2^*}(\phi_L^{\rightarrow}(f),r,s)$, $f \in L^X$, $r \in L_0$, *and* $s \in L_1$.

(2) $\mathcal{C}_{\mathcal{S}_1,\mathcal{S}_1^*}(\phi_L^{\leftarrow}(g),r,s) \leqslant \phi_L^{\leftarrow}(\mathcal{C}_{\mathcal{S}_2,\mathcal{S}_2^*}(g,r,s))$, $g \in L^Y$, $r \in L_0$, *and* $s \in L_1$.

(3) $\phi : (X,\mathcal{T}_{\mathcal{S}_1},\mathcal{T}^*_{\mathcal{S}_1^*}) \to (Y,,\mathcal{T}_{\mathcal{S}_2},\mathcal{T}^*_{\mathcal{S}_2^*})$ *is fuzzy continuous.*

Proof. (1) Suppose there exsit $f \in L^X$, $r \in L_0$, and $s \in L_1$ such that

$$\Psi\phi_L^{\rightarrow}(\mathcal{C}_{\mathcal{S}_1,\mathcal{S}_1^*}(f,r,s) \not\leqslant \mathcal{C}_{\mathcal{S}_2,\mathcal{S}_2^*}(\phi_L^{\rightarrow}(f),r,s).$$

There exists $(\zeta,\zeta^*) \in \Upsilon_X$ with $\mathcal{S}_2(\zeta,\zeta^*) \geqslant r, \mathcal{S}_2^*(\zeta,\zeta^*) \leqslant s, \zeta(\phi_L^{\rightarrow}(x),g) \perp$ and $\zeta^*(\phi_L^{\rightarrow}(x),g) \leqslant \top$ such that $\mathcal{C}_{\mathcal{S}_2,\mathcal{S}_2^*}(\phi_L^{\rightarrow}(f),r,s) \leqslant g$. On the other hand, ϕ is syntopogenous continuous, for each $(\zeta,\zeta^*) \in {}_Y$, there exsits $(\zeta,\zeta^*) \in {}_X$, with $\eta(\phi_L^{\leftarrow}(\phi_L^{\rightarrow}(f)),\phi_L^{\leftarrow}(g)) \geqslant \zeta(\phi_L^{\rightarrow}(f),g)$ and $\eta^*(\phi_L^{\leftarrow}(\phi_L^{\rightarrow}(f)),\phi_L^{\leftarrow}(g)) \leqslant \zeta^*(\phi_L^{\rightarrow}(f),g)$ such that $\mathcal{S}_1(\eta,\eta^*) \geqslant \mathcal{S}_2(\zeta,\zeta^*) \geqslant r$ and $\mathcal{S}_1^*(\eta,\eta^*) \leqslant \mathcal{S}_2^*(\zeta,\zeta^*) \leqslant s$.

It implies $\mathcal{C}_{\mathcal{S}_1,\mathcal{S}_1^*}(f,r,s) \leqslant \phi_L^{\leftarrow}(g)$. It is a contradiction. □

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Journal of the Association of Arab Universities for Basic and Applied Sciences

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ORIGINAL ARTICLE

Solving the class equation $x^d = \beta$ in an alternating group for each $\beta \in H \cap C^\alpha$ and $n \notin \theta$

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Available online 5 October 2011

KEYWORDS

Alternating groups;
Permutations;
Conjugate classes;
Cycle type;
Ambivalent groups

Abstract In this paper we find the solutions to the class equation $x^d = \beta$ in the alternating group A_n (i.e. find the solutions set $X = \{x \in A_n \mid x^d \in A(\beta)\}$) and find the number of these solutions $|X|$ for each $\beta \in H \cap C^\alpha$ and $n \notin \theta$, where $H = \{C^\alpha \text{ of } S_n \mid n > 1$, with all parts α_k of α different and odd$\}$. C^α is conjugacy class of S_n and form class C^α depends on the cycle partition α of its elements. If $(14 > n \notin \theta \cup \{9, 11, 13\})$ and $\beta \in H \cap C^\alpha$ in A_n, then F_n contains C^α, where $F_n = \{C^\alpha \text{ of } S_n \mid$ the number of parts α_k of α with the property $\alpha_k \equiv 3 \pmod 4$ is odd$\}$. In this work we introduce several theorems to solve the class equation $x^d = \beta$ in the alternating group A_n where $\beta \in H \cap C^\alpha$ and $n \notin \theta$ and we find the number of the solutions for n to be: (i) $14 > n \notin \theta$, (ii) $14 > n \notin \theta$ and $(n + 1) \notin \theta$, (iii) $14 > n \notin \theta$ and $C^\alpha \neq [1, 3, 7]$, (iv) $n = 9, 11, 13$, (v) $n > 14$.

1. Introduction

The main purpose of this work is to solve and find the number of solutions to the class equation $x^d = \beta$ in an alternating group, where β ranges over the conjugacy class $A(\beta)$ in A_n and d is a positive integer. In this paper we solve the class equation $x^d = \beta$ in A_n, where $\beta \in H \cap C^\alpha$ and $n \notin \theta = \{1, 2, 5, 6, 10, 14\}$ and we find the number of solutions when $H = \{C^\alpha \text{ of } S_n \mid n > 1$, with all parts α_k and α different and odd$\}$. C^α is conjugacy class of S_n. If $\lambda \in C^\alpha$ and $\lambda \notin H \cap C^\alpha$, then C^α does not split into the two classes $C^{\alpha\pm}$ of A_n. The Frobenius equation $x^d = c$ in finite groups was introduced by Frobenius (1903), and studied by many others, such as Ishihara et al. (2001), Takegahara (2002), Chigira (1996), who dealt with some types of finite groups, including finite cyclic groups, finite p-groups, and Wreath products of finite groups. Choose any $\beta \in S_n$ and write it as $\gamma_1\gamma_2 \ldots \gamma_{c(\beta)}$. With γ_i disjoint cycles of length α_i and $c(\beta)$ are the number of disjoint cycle factors including the 1-cycle of β. Since disjoint cycles commute, we can assume that $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{c(\beta)}$ (Rotman, 1995). Therefore $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{c(\beta)})$ is a partition of n and it is call cycle type of β. Let $C^\alpha \subset S_n$ be the set of all elements with cycle type α, then we can determine the conjugate class of $\beta \in S_n$ by using cycle type of β, since each pair of λ and β in S_n are conjugate if they have the same cycle type (Zeindler, 2010). Therefore, the number of conjugacy classes of S_n is the number of partitions of n. However, this is not necessarily true in an alternating group. Let $\beta = (1\,2\,4)$ and $\lambda = (1\,4\,2)$ be two permutations in S_4 that belong to the same conjugacy class $C^\alpha = [1, 3]$ in S_4 (i.e. $C^\alpha(\beta) = C^\alpha(\lambda)$). Since $\alpha(\beta) = (\alpha_1(\beta), \alpha_2(\beta)) = (1, 3) = (\alpha_1(\lambda), \alpha_2(\lambda)) = \alpha(\lambda)$, they have the same cycle type, but λ and β are not conjugate in A_4. Let $\beta = (1\,2\,3)(4\,5\,6)(7\,8\,9)$ and $\lambda = (5\,3\,7)(1\,6\,9)(2\,4\,8)$ in S_9 where

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Peer review under responsibility of University of Bahrain.
doi:10.1016/j.jaubas.2011.06.006

they belong to the same conjugacy class $C^\alpha = [3^3]$ in S_4 since $\alpha(\beta) = (3,3,3) = \alpha(\lambda)$. But here λ and β are conjugate in A_9. The first and second examples demonstrate that it is not necessary for two permutations to have the same cycle structure in order to be conjugate in A_n. In this work we discuss the conjugacy classes in an alternating group and we denote conjugacy class of β in A_n by $A(\beta)$.

Definition 1.1. A partition α is a sequence of nonnegative integers $(\alpha_1, \alpha_2, \ldots)$ with $\alpha_1 \geqslant \alpha_2 \geqslant \cdots$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$. The length $l(\alpha)$ and the size $|\alpha|$ of α are defined as $l(\alpha) = Max\{i \in N; \alpha_i \neq 0\}$ and $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$. We set $\alpha \vdash n = \{\alpha \text{ partition}; |\alpha| = n\}$ for $n \in N$. An element of $\alpha \vdash n$ is called a partition of n (Zeindler, 2010).

Remark 1.2. We only write the non zero components of a partition. Choose any $\beta \in S_n$ and write it as $\gamma_1\gamma_2 \ldots \gamma_{c(\beta)}$. With γ_i disjoint cycles of length α_i and $c(\beta)$ are the number of disjoint cycle factors including the 1-cycle of β. Since disjoint cycles commute, we can assume that $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_{c(\beta)}$. Therefore $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{c(\beta)})$ is a partition of n and each α_i is called part of α (see Zeindler, 2010).

Definition 1.3. We call the partition $\alpha = \alpha(\beta) = (\alpha_1(\beta), \alpha_2(\beta), \ldots, \alpha_{c(\beta)}(\beta))$ the cycle type of β (Zeindler, 2010).

Definition 1.4. Let α be a partition of n. We define $C^\alpha \subset S_n$ to be the set of all elements with cycle type α (Zeindler, 2010).

Definition 1.5. Let $\beta \in S_n$ be given. We define $c_m = c_m^{(n)} = c_m^{(n)}(\beta)$ to be the number of cycles of length m of β (Zeindler, 2010).

Lemma 1.6. *$C^{\alpha\pm}$ of A_n are ambivalent if and only if the number of parts α_k of α with the property $\alpha_k \equiv 3 \pmod 4$ is even (James et al., 1984).*

Remark 1.7

(**1**) The relationship between partitions and c_m is as follows: if $\beta \in C^\alpha$ is given then $c_m^{(n)}(\beta) = |\{i : \alpha_i = m\}|$ (see Zeindler (2010)).
(**2**) The cardinality of each C^α can be found as follows: $|C^\alpha| = \frac{n!}{z_\alpha}$ with $z_\alpha = \prod_{r=1}^{n} r^{c_r}(c_r)!$ and $c_r = c_r^{(n)}(\beta) = |\{i : \alpha_i = r\}|$ (see Bump (2004)).
(**3**) If x is a solution of $x^d = \beta$, d is a positive integer, and y is a conjugate to x, then y is a solution of $x^d = \lambda$, where λ is conjugate to β in an alternating group (or any finite group). We call $x^d = \beta$ a class equation in A_n, where β and x belong to conjugate classes in an alternating group (see Taban (2007)).

Definition 1.8. Let $\beta \in C^\alpha$, where β is a permutation in an alternating group. We define the $A(\beta)$ conjugacy class of β in A_n by:

$$A(\beta) = \{\gamma \in A_n \mid \gamma = t\beta t^{-1};\ \text{for some } t \in A_n\} = \begin{cases} C^\alpha, & (\text{if } \beta \notin H) \\ C^{\alpha+} \text{ or } C^{\alpha-}, & (\text{if } \beta \in H) \end{cases}$$

where $H = \{C^\alpha \text{ of } S_n | n > 1$, with all parts α_k of α different and odd$\}$.

Remark 1.9

(1) $\beta \in C^\alpha \cap H^C \cap A_n \Rightarrow A(\beta) = C^\alpha$, where H^C is a complement set of H.
(2) $\beta \in C^\alpha \cap H \Rightarrow \beta \in A_n$ and C^α splits into the two classes $C^{\alpha\pm}$ of A_n.
(3) $\beta \in H \Rightarrow A(\beta) = \begin{cases} C^{\alpha+} \text{ if } \beta \in C^{\alpha+} \\ C^{\alpha-} \text{ if } \beta \in C^{\alpha-} \end{cases}$
(4) If $n \in \theta = \{1,2,5,6,10,14\}$, then for each $\beta \in A_n$ we have β conjugate to its inverse in $A_n (\beta \underset{A_n}{\approx} \beta^{-1})$.

Definition 1.10. Let $F_n = \{C^\alpha \text{ of } S_n |$ the number of parts α_k of α with the property $\alpha_k \equiv 3 \pmod 4$ is odd$\}$. Then for each $\beta \in H \cap C^\alpha \cap F_n$ in S_n, we define $C^{\alpha\pm}$ of A_n by:

$$C^{\alpha+} = \{\lambda \in A_n \mid \lambda = \gamma\beta\gamma^{-1};\ \text{for some } \gamma \in A_n\} = A(\beta)$$
$$C^{\alpha-} = \{\lambda \in A_n \mid \lambda = \gamma\beta^{-1}\gamma^{-1};\ \text{for some } \gamma \in A_n\} = A(\beta^{-1})$$

Remark 1.11

(1) Suppose $n \notin \theta$ & $\beta \in H \cap C^\alpha$ in A_n, then we have:
  (i) If $(n+1) \in \theta$, then $C^\alpha \neq [4]$ (since $H \cap [4] = \phi$).
  (ii) If $(n+1) \in \theta$, and $C^\alpha \neq [n]$, then β does not conjugate to its inverse in A_n.
  (iii) If $(n+1) \in \theta$ and $C^\alpha = [n]$, then $n = (9 \text{ or } 13)$, and $(\beta \underset{A_n}{\approx} \beta^{-1})$. So we define $C^{\alpha\pm}$ by: $C^{\alpha+} = [n]^+ = A(\beta)$ and $C^{\alpha-} = \{[n] - A(\beta)\}$ or $\{A(\beta^\#)$; for some $\beta^\# \in [n]$ do not conjugate to $\beta\}$.
(2) Suppose $n \in \theta$ and $\beta \in H \cap C^\alpha$ in A_n, where $C^\alpha = [k_1, k_2, \ldots, k_L]$ and $k_i \neq 1$, $(1 \leqslant i \leqslant L)$ then

  we have:

  (**i**) $\beta \in H \cap [1, k_1, k_2, \ldots, k_L]$ in A_{n+1}.

  (**ii**) $(\beta \underset{A_{n+1}}{\approx} \beta^{-1})$.

*Finally, based on (1) and (2), we consider for all $(14 > n \notin \theta)$ and $\beta \in H \cap C^\alpha$ in A_n, but $\beta \notin H \cap C^\alpha \cap F_n$ in $A_n \Rightarrow C^\alpha = [9]$ or $[13]$ or $[1,3,7]$. So we define $[1,3,7]^{\mp}$ by $[1,3,7]^+ = A(\beta)$ and $[1,3,7]^- = A(\beta^\#)$, where $\beta^\# \in [1,3,7]$ does not conjugate to β.

Theorem 1.12. *Let $A(\beta)$ be the conjugacy class of β in A_n, $\beta \in [K,L] \cap H$ and $14 > n \notin \theta$, where $[K,L]$ is a class of S_n. If p is a positive integer such that $\gcd(p,K) = 1$ and $\gcd(p,L) = 1$, then the solutions of $x^p \in A(\beta)$ in A_n are:*

(1) $[K,L]^-$ if $\beta^p = (\beta^{-1}$ or $\gamma)$, where γ is conjugate to β^{-1}.
(2) $[K,L]^+$ if $\beta^p = (\beta$ or $\gamma)$, where γ is conjugate to β.

Proof. Since $\beta \in A_n \cap [K,L] \cap H$, $[K,L]$ splits into two classes $[K,L]^\pm$ of A_n. However, $14 > n \notin \theta \Rightarrow [K,L] \in F_n \Rightarrow A(\beta) = [K,L]^+$ and $A(\beta^{-1}) = [K,L]^{-1}$. Also, since $\gcd(p,K) = 1$, p does not divide K, and since $\gcd(p,L) = 1$, p does not divide L. Then by Taban (2007, lemma 3.9) we have $[K,L]$ as the solution set of $x^p \in [K,L]$ in S_n.

(1) Assume $\beta^p = (\beta^{-1}$ or $\gamma = b\beta^{-1}b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [K,L]$. Then either $\lambda \in [K,L]^+$, or $\lambda \in [K,L]^-$.

If $\lambda \in [K,L]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$, $\lambda^p = t\beta^p t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix}$, $\lambda^p \in [K,L]^- \Rightarrow \lambda^p \notin [K,L]^+ = A(\beta)$.

If $\lambda \in [K,L]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$, $\lambda^p = t\beta^{-p}t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix}$, $\lambda^p \in [K,L]^+ = A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in is $[K,L]^-$.

(2) Assume $\beta^p = (\beta$ or $\gamma = b\beta b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [K,L]$. Then either $\lambda \in [K,L]^+$ or $\lambda \in [K,L]^-$.

If $\lambda \in [K,L]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$, $\lambda^p = t\beta^p t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix}$, $\lambda^p \in [K,L]^+ = A(\beta) \Rightarrow \lambda^p \notin [K,L]$.

If $\lambda \in [K,L]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$, $\lambda^p = t\beta^{-p}t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix}$, $\lambda^p \in [K,L]^- \Rightarrow \lambda^p \notin A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in A_n is $[K,L]^+$.

Lemma 1.13. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $\beta \in [n] \cap H$, where $[n]$ is a class of S_n. If p and q are two different prime numbers, $p \mid n$ and $q \mid n$, then there is no solution of $x^{pq} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.9) there is no solution of $x^{pq} \in [n]$ in S_n. So there is no solution of $x^{pq} \in A(\beta)$ in A_n.

Lemma 1.14. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $\beta \in [n] \cap H$, where $[n]$ is a class of S_n. If p and q are two different prime numbers, $p \mid n$ and q does not divide n, then there is no solution of $x^{pq} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.4) there is no solution of $x^{pq} \in [n]$ in S_n. So there is no solution of $x^{pq} \in A(\beta)$ in A_n.

Theorem 1.15. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $\beta \in [n] \cap H$, where $[n]$ is a class of S_n. If p and q are different prime numbers such that $gcd(p,n) = 1$ and $gcd(q,n) = 1$, then the solutions of $x^{pq} \in A(\beta)$ in A_n are:*

(1) $[n]^-$ if $\beta^{pq} = (\beta^{-1}$ or $\gamma)$, where γ is conjugate to β^{-1}.
(2) $[n]^+$ if $\beta^{pq} = (\beta$ or $\gamma)$, where γ is conjugate to β.

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$ and $A(\beta^{-1}) = [n]^-$. Also, since, $gcd(p,n) = 1$, p does not divide n, and $gcd(q,n) = 1$, q does not divide n. Then by Taban (2007, lemma 3.4) we have $[n]$ as the solution set of $x^{pq} \in [n]$ in S_n.

(1) Assume $\beta^{pq} = (\beta^{-1}$ or $\gamma = b\beta^{-1}b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$, $\lambda^{pq} = t\beta^{pq}t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix}$, $\lambda^{pq} \in [n]^- \Rightarrow \lambda^{pq} \notin [n]^+ = A(\beta)$.

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$, $\lambda^{pq} = t\beta^{-pq}t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix}$, $\lambda^{pq} \in [n]^+ = A(\beta)$. Then the solution set of $x^{pq} \in A(\beta)$ in A_n is $[n]^-$.

(2) Assume $\beta^{pq} = (\beta$ or $\gamma = b\beta b^{-1})$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$, $\lambda^{pq} = t\beta^{pq}t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix}$, $\lambda^{pq} \in [n]^+ = A(\beta) \Rightarrow \lambda^{pq} \notin [n]^-$.

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$, $\lambda^{pq} = t\beta^{-pq}t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix}$, $\lambda^{pq} \in [n]^- \Rightarrow \lambda^{pq} \notin A(\beta)$. Then the solution set of $x^{pq} \in A(\beta)$ in A_n is $[n]^+$.

Lemma 1.16. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $\beta \in [n] \cap H$, where $[n]$ is a class of S_n. If p is prime number such that $p \mid n$, then there is no solution of $x^p \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [K,L] \cap H$, $[K,L]$ splits into two classes $[K,L]^{\pm}$ of A_n. However, $14 > n \notin \theta \Rightarrow [K,L] \in F_n \Rightarrow A(\beta) = [K,L]^+$, then by Taban (2007, lemma 3.8) we have there is no solution of $x^p \in [K,L]$ in S_n. So there is no solution of $x^p \in A(\beta)$ in A_n.

Theorem 1.17. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$, and $\beta \in [K,L] \cap H$ where $[K,L]$ is a class of S_n. If $p \mid K$ and $p \mid L$, then there is no solution of $x^p \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [K,L] \cap H$, $[K,L]$ splits into two classes $[K,L]^{\pm}$ of A_n. However, $14 > n \notin \theta \Rightarrow [K,L] \in F_n \Rightarrow A(\beta) = [K,L]^+$, then by Taban (2007, lemma 3.8) we have there is no solution of $x^p \in [K,L]$ in S_n. So there is no solution of $x^p \in A(\beta)$ in A_n.

Theorem 1.18. *Let $A(\beta)$ be the conjugacy class of β in A_n, $\beta \in [n] \cap H$, $14 > n \notin \theta$ & $(n+1) \notin \theta$, where $[n]$ is a class of S_n. If p is a prime number such that $gcd(n,p) = 1$, then the solutions of $x^p \in A(\beta)$ in A_n are:*

(1) $[n]^-$ if $\beta^P = (\beta^{-1}$ or $\gamma)$, where γ is conjugate to β^{-1}.
(2) $[n]^+$ if $\beta^P = (\beta$ or $\gamma)$, where γ is conjugate to β.

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$ and $A(\beta^{-1}) = [n]^-$. Also, since $gcd(n,p) = 1$, p does not divide n. Then by Taban (2007, lemma 3.2) we have $[n] = [n]^+ \cup [n]^-$ as a solution of $x^p \in [n]$ in S_n. But, $[n] = A(\beta) \cup [n]^{-1}$ then the solution set of $x^p \in A(\beta)$ in A_n is either $[n]^-$ or $[n]^+$.

(1) Assume $\beta^p = (\beta^{-1}$ or $\gamma = b\beta^{-1}b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix}, \quad \lambda^p \in [n]^- \Rightarrow \quad \lambda^p \notin [n]^+ = A(\beta),$$

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix}, \quad \lambda^p \in [n]^+ = A(\beta).$$

Then the solution set of $x^p \in A(\beta)$ in A_n is $[n]^-$.

(2) Assume $\beta^p = (\beta$ or $\gamma = b\beta b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix}, \quad \lambda^p \in [n]^+ = A(\beta) \Rightarrow \quad \lambda^p \notin [n]^-,$$

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix}, \quad \lambda^p \in [n]^- \Rightarrow \quad \lambda^p \notin A(\beta).$$

Then the solution set of $x^p \in A(\beta)$ in A_n is $[n]^+$.

Example 1.19. Find the solutions of $x^5 \in A(1\ 3\ 2)$ in A_3.

Solution:

Since $(14 > 3 \notin \theta)$, $(1\ 3\ 2) \in [3] \cap H$, $gcd(3,5) = 1$, and $(1\ 3\ 2)^5 = (1\ 2\ 3) = (1\ 3\ 2)^{-1}$, then the solution of $x^5 \in A(1\ 3\ 2)$ in A_3 is $[3]^- = \{(1\ 2\ 3)\}$.

Lemma 1.20. *Let $A(\beta)$ be the conjugacy class of β in A_n, and $\beta \in [n] \cap H$, where $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $[n]$ is a class of S_n. If $p \mid n$, then there is no solution of $x^{p^m} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.12) we have there is no solution of $x^{p^m} \in [n]$ in S_n. So there is no solution of $x^{p^m} \in A_{(\beta)}$ in A_n.

Lemma 1.21. *Let $A(\beta)$ be the conjugacy class of β in A_n, and $\beta \in [n] \cap H$, where $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $[n]$ is a class of S_n. If p and q are different prime numbers, $p \mid n$ and $q \mid n$ then there is no solution of $x^{p^m q^d} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.14) we have there is no solution of $x^{p^m q^d} \in [n]$ in S_n. So there is no solution of $x^{p^m q^d} \in A_\beta$ in A_n.

Lemma 1.22. *Let $A(\beta)$ be the conjugacy class of β in A_n, If p and q are two different prime numbers such that $p \mid n$, q does not divide n, and $\beta \in [n] \cap H$ where $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $[n]$ is a class of S_n, then there is no solution of $x^{p^m q^d} \in A(\beta)$ in A_n.*

Proof. Since$\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.15) we have there is no solution of $x^{p^m q^d} \in [n]$ in S_n. So there is no solution of $x^{p^m q^d} \in A(\beta)$ in A_n.

Lemma 1.23. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, where $[n]$ is a class of S_n. If p and q are different prime numbers such that $gcd(p,n) = 1$ and $gcd(q,n) = 1$, then the solutions of $x^{p^m q^d} \in A(\beta)$ in A_n are:*

(1) $[n]^-$ if $\beta^{p^m q^d} = (\beta^{-1}$ or $\gamma)$, where γ is conjugate to β^{-1}.
(2) $[n]^+$ if $\beta^{p^m q^d} = (\beta$ or $\gamma)$, where γ is conjugate to β.

Proof. $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, and since $gcd(p,n) = 1$, p does not divide n and $gcd(q,n) = 1 \Rightarrow q$ does not divide n. Then by Taban (2007, lemma 3.2) we have $[n] = [n]^+ \cup [n]^-$ as a solution set of $x^{p^m q^d} \in [n]$ in S_n. But, $[n] = A(\beta) \cup [n]^-$ then the solution set of $x^{p^m q^d} \in A(\beta)$ in A_n is either $[n]^-$ or $[n]^+$.

(1) Assume $\beta^{p^m q^d} = (\beta^{-1}$ or $\gamma = b\beta^{-1}b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^{p^m q^d} = t\beta^{(p^m q^d)}t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix},$$

$$\lambda^{p^m q^d} \in [n]^- \Rightarrow \lambda^{p^m q^d} \notin [n]^+ = A(\beta),$$

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^{p^m q^d} = t\beta^{-(p^m q^d)}t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix}, \quad \lambda^{p^m q^d} \in [n]^+ = A(\beta).$$

Then the solution set of $x^{p^m q^d} \in A(\beta)$ in A_n is $[n]^-$.

(2) Assume $\beta^{p^m q^d} = (\beta$ or $\gamma = b\beta b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [n]$. Then either $\lambda \in [n]^+$ or $\lambda \in [n]^-$.

If $\lambda \in [n]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^{p^m q^d} = t\beta^{p^m q^d}t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix},$$

$$\lambda^{p^m q^d} \in [n]^+ = A(\beta) \Rightarrow \lambda^{p^m q^d} \notin [n]^-,$$

If $\lambda \in [n]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^{p^m q^d} = t\beta^{-(p^m q^d)}t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix},$$

$\lambda^{p^m q^d} \in [n]^- \Rightarrow \lambda^{p^m q^d} \notin A(\beta)$.

Then the solution set of $x^{p^m q^d} \in A(\beta)$ in A_n is $[n]^+$.

Lemma 1.24. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$ & $(n+1) \notin \theta$, and $\beta \in [n] \cap H$, where $[n]$ is a class of S_n. If $p_1, p_2, \ldots, p_m$ are different prime numbers such that, $p_i \mid n$, $\forall i = 1, \ldots, m$, then there is no solution for $x^{p_1^{d_1} p_2^{d_2} \cdots p_m^{d_m}} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap [n] \cap H$, $[n]$ splits into two classes $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.17) we have there is no solution of $x^{p_1^{d_1} p_2^{d_2} \cdots p_m^{d_m}} \in [n]$ in S_n. So there is no solution of $x^{p_1^{d_1} p_2^{d_2} \cdots p_m^{d_m}} \in A(\beta)$ in A_n.

Lemma 1.25. *Let $A(\beta)$ be the conjugacy class of β in A_n, $\beta \in [n] \cap H$, and $14 > n \notin \theta$ & $(n+1) \notin \theta$, where is $[n]$ is a class of S_n. If $p_1, p_2, \ldots, p_m$ are different prime numbers and $p \mid n$ for some i, then there is no solution for $x^{p_1^{d_1} p_2^{d_2} \cdots p_m^{d_m}} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap H$, $[n]$ splits into two classes of $[n]^{\pm}$ of A_n. However, $14 > n \notin \theta$ & $(n+1) \notin \theta \Rightarrow [n] \in F_n \Rightarrow A(\beta) = [n]^+$, then by Taban (2007, lemma 3.19) we have there is no solution of $x^{p_1^{d_1} p_2^{d_2} \cdots P_m^{d_m}} \in [n]$ in S_n. So there is no solution of $x^{p_1^{d_1} p_2^{d_2} \cdots P_m^{d_m}} \in A(\beta)$ in A_n.

Lemma 1.26. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$, and $\beta \in [k_1, k_2, \ldots, k_l] \cap H$, where $[k_1, k_2, \ldots, k_l] \neq [1,3,7]$ is a class of S_n. If p is a prime number such that $gcd(p, k_i) = 1$, for each i, then the solutions of $x^p \in A(\beta)$ are:*

(1) $[k_1, k_2, \ldots, k_l]^-$ if $\beta^p = \beta^{-1}$ or γ, where γ is conjugate to β^{-1}.

(2) $[k_1, k_2, \ldots, k_l]^+$ if $\beta^p = \beta$ or γ, where γ is conjugate to β.

Proof. Since $\beta \in A_n \cap H \cap [k_1, k_2, \ldots, k_l]$, $[k_1, k_2, \ldots, k_l]$ splits into two classes $[k_1, k_2, \ldots, k_l]^{\pm}$ of A_n. However, $14 \geqslant n \notin \theta$ and $[k_1, k_2, \ldots, k_l] \neq [1,3,7] \Rightarrow [k_1, k_2, \ldots, k_l] \in F \Rightarrow A(\beta) = [k_1, k_2, \ldots, k_l]^+$, and $A(\beta^{-1}) = [k_1, k_2, \ldots, k_l]^-$. Also, since $gcd(p, k_i) = 1$ for each $\lambda[k_1, k_2, \ldots, k_l] \Rightarrow \lambda^p[k_1, k_2, \ldots, k_l]$. Then by Taban (2007, lemma 2.8) we have for each $\lambda \in [k_1, k_2, \ldots, k_l] \Rightarrow \lambda^p \in [k_1, k_2, \ldots, k_l]$.

(1) Assume $\beta^p = (\beta^{-1}$ or $\gamma = b\beta^{-1}b^{-1}$; for some $b \in A_n)$, and let $\lambda \in [k_1, k_2, \ldots, k_l]$. Then either $\lambda \in [k_1, k_2, \ldots, k_l]^+$ or $\lambda \in [k_1, k_2, \ldots, k_l]^-$.

If $\lambda \epsilon [k_1, k_2, \ldots, k_l]^+$, $\exists t \epsilon A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix},$$

$\lambda^p \epsilon [k_1, k_2, \ldots, k_1]^- \Rightarrow \lambda^p \notin [k_1, k_2, \ldots, k_1]^+ = A(\beta)$,

If $\lambda \epsilon [k_1, k_2, \ldots, k_l]^-$, $\exists t \epsilon A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix},$$

$\lambda^p \epsilon [k_1, k_2, \ldots, k_1]^+ = A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in A_n is $[k_1, k_2, \ldots, k_1]^-$.

(2) Assume $\beta^p = (\beta$ *or* $\gamma = b\beta b^{-1}$ for some $b \in A_n)$, and let $\lambda \in [k_1, k_2, \ldots, k_1]$. Then either $\lambda \in [k_1, k_2, \ldots, k_1]^+$ or $\lambda \in [k_1, k_2, \ldots, k_1]^-$.

If $\lambda \epsilon [k_1, k_2, \ldots, k_l]^+$, $\exists t \epsilon A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix},$$

$\lambda^p \epsilon [k_1, k_2, \ldots, k_1]^+ = A(\beta) \Rightarrow \lambda^p \notin [k_1, k_2, \ldots, k_l]^-$,

If $\lambda \epsilon [k_1, k_2, \ldots, k_l]^-$, $\exists t \epsilon A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix},$$

$\lambda^p \epsilon [k_1, k_2, \ldots, k_1]^- \Rightarrow \lambda^p \notin A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in A_n is $[k_1, k_2, \ldots, k_1]^+$.

Remark 1.27. If there is no solution for $x^p \in [k_i]$, for some $1 \leqslant i \leqslant l$ then there exists no solution for $x^p \in [k_1, k_2, \ldots, k_l]$.

Lemma 1.28. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$, and $\beta \in [k_1, k_2, \ldots, k_l] \cap H$, where $[k_1, k_2, \ldots, k_l] \neq [1,3,7]$ is a class of S_n. If p is a prime number such that $p \mid k_i$, for some i, then no solution of $x^p \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap H \cap [k_1, k_2, \ldots, k_l]$, $[k_1, k_2, \ldots, k_l]$ splits into two classes $[k_1, k_2, \ldots, k_l]^{\pm}$ of A_n. Since $A(\beta) = [k_1, k_2, \ldots, k_l]^+$ and $p \mid k_i$, then by Taban (2007, lemma 3.1) we have no solution for $x^p \in [k_i]$ in S_n. Then no solution for $x^p \in [k_1, k_2, \ldots, k_l]$ in. So no solution for $x^p \in A(\beta)$ in A_n.

Lemma 1.29. *Let $A(\beta)$ be the conjugacy class of β in A_n, $14 > n \notin \theta$, and $\beta \in [k_1, k_2, \ldots, k_l] \cap H$, where $[k_1, k_2, \ldots, k_l] \neq [1,3,7]$ is a class of S_n, P is a prime number, is a positive integer. If for some $1 \leqslant i \leqslant l$ such that $p \mid k_i$, then we have no solution of $x^{p^m} \epsilon A\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap H \cap [k_1, k_2, \ldots, k_l]$, $[k_1, k_2, \ldots, k_l]$ splits into two classes $[k_1, k_2, \ldots, k_l]^{\pm}$ of A_n. Since $A(\beta) = [k_1, k_2, \ldots, k_l]^+$ and $p \mid k_i$, then by Taban (2007, lemma 3.12) we have no solution for $x^{p^m} \epsilon [k_i]$ in S_n. So no solution for $x^{p^m} \epsilon [k_1, k_2, \ldots, k_l]$ in S_n. Then no solution for $x^{p^m} \epsilon A(\beta)$ in A_n.

Definition 1.30. Let $\beta \in [9]$ of S_9, where $\beta = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$. We define class $[9]^+$ of A_9 by $A(\beta) = [9]^+ = \{\mu \in [9] \mid \mu = t\beta t^{-1}$; for some $t \in A_9\}$.

Remark 1.31

(i) $[9]^{-1} = [9] - A(\beta) = \{\mu \in [9] \mid \mu \neq t\beta\, t^{-1}$; *for all* $t \in A_9\}$.

(ii) Let $\beta \in [9]$ of S_9 where $\beta = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$, $\bar{\beta} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ $\bar{\bar{\beta}} = (a_1, a_5, a_9, a_4, a_8, a_3, a_7, a_2, a_6)$ and d is a positive integer we have:

(1)	$\beta^d = \beta \iff d \equiv 1 \pmod 9$
(2)	$\beta^d = \bar{\beta} \iff d \equiv 2 \pmod 9$
(3)	$\beta^d = \bar{\bar{\beta}} \iff d \equiv 4 \pmod 9$
(4)	$\beta^d = \bar{\beta}^{-1} \iff d \equiv 5 \pmod 9$
(5)	$\beta^d = \bar{\beta}^{-1} \iff d \equiv 7 \pmod 9$
(6)	$\beta^d = \beta^{-1} \iff d \equiv 8 \pmod 9$

(iii) (1) $A(\beta) = A(\beta^{-1})$, $A(\bar{\beta}) = A(\bar{\bar{\beta}}^{-1})$, $A(\bar{\bar{\beta}}) = A(\overline{\overline{\beta^{-1}}})$ [Since for each $\lambda \in [9]^{\mp}$ in A_9, where $\lambda = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9)$, $\exists \mu = (b_1, b_8)$ (b_2, b_7) (b_3, b_6) $(b_4, b_5) \in A_9$ such that $\mu\lambda\mu^{-1} = \lambda^{-1}$].

(2) $A(\beta) = A(\bar{\beta})$ since $\exists t = (a_3, a_6)$ $(a_1, a_5, a_7, a_8, a_4, a_2) \in A_9$ such that $t\beta t^{-1} = \bar{\beta}$.

(3) $A(\beta^{-1}) = A(\bar{\bar{\beta}})$ since $\exists t = (a_3, a_9)$ $(a_1, a_8, a_7, a_2, a_4, a_5) \in A_9$ such that $t\bar{\bar{\beta}}t^{-1} = \beta^{-1}$.

Theorem 1.32. *Let $\beta \in [9]$ of S_9. If d is a positive integer such that $gcd(d,9) = 1$, then the solutions of $x^d \epsilon A(\beta)$ in A_9 are $A(\beta)$.*

Proof. Since $\beta \in [9] \cap H \cap A_9$, $[9]$ splits into two classes $A(\beta)$ & $[9]^-$ of A_9 and $gcd(d,9) = 1$, then d does not divide 9. Then by Taban (2007, lemma 3.2), the solution set of $x^d \in [9]$ in S_9 is $[9]$. For each $\lambda \in [9]$ we have $\alpha \in A(\beta)$ *or* $\alpha \notin A(\beta)$. If, $\alpha \in A(\beta)$ we have $\lambda \underset{A_9}{\approx} \beta (\lambda$ conjugate to β in $A_9) \Rightarrow \lambda^d \underset{A_9}{\approx} \beta^d$. How-ever, $\beta^d \underset{A_9}{\approx} \beta \Rightarrow \lambda^d \underset{A_9}{\approx} \beta \Rightarrow \lambda^d \in A(\beta)$. If $\alpha \notin A(\beta)$, assume $\lambda^d \in A(\beta) \Rightarrow \lambda^d \underset{A_9}{\approx} \beta$. But $\beta \underset{A_9}{\approx} \beta^d \Rightarrow \lambda^d \underset{A_9}{\approx} \beta^d \Rightarrow \lambda \underset{A_9}{\approx} \beta \Rightarrow$, (which is a contradiction). Then the solution of in A_9 is $A(\beta)$.

Definition 1.33. Let $\beta = \gamma\lambda \in [1,3,7]$ of S_{11}, where $\gamma = (b_1, b_2, b_3)$, $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$. We define classes $[1,3,7]^{\pm}$ of A_{11} by:

$A(\beta) = [1,3,7]^+ = \{\mu \in [1,3,7] \mid \mu = t\beta t^{-1}$; *for some* $t \in A_{11}\}$ and $A(\overset{\#}{\beta}) = [1,3,7]^- = \{\mu \in [1,3,7] \mid \mu = t\overset{\#}{\beta}t^{-1}$; *for some* $t \in A_{11}\}$ where $\overset{\#}{\beta} = \gamma\bar{\lambda}$ and $\bar{\lambda} = (a_1, a_4, a_7, a_3, a_6, a_2, a_5)$.

Remark 1.34

(i) Let $\beta = \gamma\lambda \in [1,3,7]$ of S_{11} where $\gamma = (b_1, b_2, b_3)$, $\underline{\lambda} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$, $\bar{\lambda} = (a_1, a_4, a_7, a_3, a_6, a_2, a_5)$, $\bar{\bar{\lambda}} = (a_1, a_3, a_5, a_7, a_2, a_4, a_6)$, and d is a positive integer number. We have:

(1)	$\beta^d = \beta \iff d \equiv 1 \pmod{21}$
(2)	$\beta^d = \gamma^{-1}\bar{\bar{\lambda}} \iff d \equiv 2 \pmod{21}$
(3)	$\beta^d = \gamma\bar{\lambda}^{-1} \iff d \equiv 4 \pmod{21}$
(4)	$\beta^d = \gamma^{-1}\bar{\bar{\lambda}}^{-1} \iff d \equiv 5 \pmod{21}$
(5)	$\beta^d = \gamma^{-1}\lambda \iff d \equiv 8 \pmod{21}$
(6)	$\beta^d = \gamma\bar{\lambda} \iff d \equiv 10 \pmod{21}$
(7)	$\beta^d = \gamma^{-1}\bar{\lambda}^{-1} \iff d \equiv 11 \pmod{21}$
(8)	$\beta^d = \gamma\underline{\lambda}^{-1} \iff d \equiv 13 \pmod{21}$
(9)	$\beta^d = \gamma\bar{\bar{\lambda}} \iff d \equiv 16 \pmod{21}$
(10)	$\beta^d = \gamma^{-1}\bar{\lambda} \iff d \equiv 17 \pmod{21}$
(11)	$\beta^d = \gamma\bar{\bar{\lambda}}^{-1} \iff d \equiv 19 \pmod{21}$
(12)	$\beta^d = \beta^{-1} \iff d \equiv 20 \pmod{21}$

(ii) (1) $A(\beta) = A(\beta^{-1})$, $A(\gamma^{-1}\bar{\bar{\lambda}}) = A(\gamma\bar{\bar{\lambda}}^{-1})$, $A(\gamma\bar{\lambda}^{-1}) = A(\gamma^{-1}\bar{\lambda})$, $A(\gamma^{-1}\bar{\bar{\lambda}}^{-1}) = A(\gamma\bar{\bar{\lambda}})$, $A(\gamma^{-1}\lambda) = A(\gamma\lambda^{-1})$, $A(\gamma\bar{\lambda}) = A(\gamma^{-1}\bar{\lambda}^{-1})$ [Since for each $\beta = \gamma\lambda \in [1,3,7]$ in A_{11} where $\gamma = (b_1, b_2, b_3)$, $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$, $\exists \mu = (b_1, b_3)$ (a_2, a_7) (a_3, a_6) $(a_4, a_5) \in A_{11}$ such that $\mu\lambda\mu^{-1} = \lambda^{-1}$].

(2) $A(\beta) = A(\gamma^{-1}\bar{\lambda})$ [since $\exists t = (b_2, b_3)$ $(a_1, a_4, a_5, a_3, a_7, a_6) \in A_{11}$ such that $t\beta t^{-1} = \gamma^{-1}\bar{\lambda}$].

(3) $A(\beta) = A(\gamma\bar{\bar{\lambda}})$ [since $\exists$ $t = (a_1, a_3, a_4)$ $(a_7, a_6, a_2) \in A_{11}$ such that $t\beta t^{-1} = \gamma\bar{\bar{\lambda}}$].

(4) $A(\gamma\bar{\bar{\lambda}}^{-1}) = A(\gamma\bar{\lambda})$ [since $\exists t = (a_1, a_4, a_2)$ $(a_3, a_5, a_6) \in A_{11}$ such that $t\gamma\bar{\bar{\lambda}}^{-1}t^{-1} = \gamma\bar{\lambda}$].

(5) $A(\gamma\bar{\bar{\lambda}}^{-1}) = A(\gamma\lambda^{-1})$ [since $\exists t = (a_1, a_6, a_5)$ $(a_2, a_3, a_7) \in A_{11}$ such that $t\gamma\lambda^{-1}t^{-1} = \gamma\bar{\bar{\lambda}}^{-1}$].

Theorem 1.35. *Let $L = \{m \in N \mid m \equiv q \pmod{21}$; for some $q = 1, 4, 5, 16, 17, 20\}$. If d is a positive integer such that $gcd(d,3) = 1$ & $gcd(d,7) = 1$ and $\beta \in [1,3,7]$ of S_{11}, then the solutions of $x^d \in A(\beta)$ in A_{11} are:*

(1) $A(\beta)$ if $d \in L$.

(2) $A(\overset{\#}{\beta})$ if $d \notin L$.

Proof. Since $\beta \in [1,3,7] \cap H \cap A_{11}$, $[1,3,7]$ splits into two classes $A(\beta)$ & $A(\overset{\#}{\beta})$ of A_{11}, $gcd(d,3) = 1$ and $gcd(d,7) = 1$, then d does not divide 3 and d does not divide 7. Then by Taban (2007, lemma 2.8) we have $[1,3,7] = A(\beta) \cup A(\overset{\#}{\beta})$ as a solution set of $x^d \in [1,3,7] = A(\beta) \cup A(\overset{\#}{\beta})$ in S_{11}. However, $A(\beta) \cap A(\overset{\#}{\beta}) = \phi$, so for each $\pi \in [1,3,7] \Rightarrow (\pi \in A(\beta)$ & $\pi \notin A(\overset{\#}{\beta}))$ or $(\pi \in A(\overset{\#}{\beta})$ & $\pi \notin A(\beta))$.

(1) Assume $d \in L$. If $\pi \in A(\beta)$, then we have $(\pi \underset{A_{11}}{\approx} \beta)$ π conjugate to β in A_{11}. However, $\pi^d \underset{A_{11}}{\approx} \pi$ (since $d \in L) \Rightarrow \pi^d \underset{A_{11}}{\approx} \beta \Rightarrow \pi^d \in A(\beta)$ & $\pi^d \notin A(\overset{\#}{\beta})$. If $\pi \in A(\overset{\#}{\beta})$, we have $(\pi \underset{A_{11}}{\approx} \overset{\#}{\beta})$. But $\pi^d \underset{A_{11}}{\approx} \pi$ (since $d \in L) \Rightarrow \pi^d \underset{A_{11}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A(\overset{\#}{\beta})$ & $\pi^d \notin A(\beta)$. Then the solution set of $x^d \in A(\beta)$ in A_{11} is $A(\beta)$.

(2) Assume $d \notin L$. If $\pi \in A(\beta)$, then we have $(\pi \underset{A_{11}}{\approx} \beta) \Rightarrow \overset{\#}{\pi} \underset{A_{11}}{\approx} \overset{\#}{\beta}$. However, $\pi^d \underset{A_{11}}{\approx} \overset{\#}{\pi}$ (since $d \notin L) \Rightarrow \pi^d \underset{A_{11}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A(\overset{\#}{\beta})$ & $\pi^d \notin A(\beta)$. If $\pi \in A(\overset{\#}{\beta}) \Rightarrow (\pi \underset{A_{11}}{\approx} \overset{\#}{\beta}) \Rightarrow \overset{\#}{\pi} \underset{A_{11}}{\approx} \overset{\#}{\beta}$. But $\pi^d \underset{A_{11}}{\approx} \overset{\#}{\pi}$ (since $d \notin L) \Rightarrow \pi^d \underset{A_{11}}{\approx} \beta \Rightarrow \pi^d \in A(\beta)$ & $\pi^d \notin \overset{\#}{\beta}$. Then the solution set of $x^d \in A(\beta)$ in A_{11} is $A(\overset{\#}{\beta})$.

Definition 1.36. Let $\beta \in [13]$ of S_{13}, where $\beta = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13})$. We define classes $[13]^{\pm}$ of A_{13} by:

$A(\beta) = [13]^+ = \{\mu \in [13] \mid \mu = t\beta t^{-1}$; *for some* $t \in A_{13}\}$ and

$A(\beta^{\#}) = [13]^- = \{\mu \in [13] \mid \mu = t\beta^{\#}t^{-1}$; for some $t \in A_{13}\}$.

where $\beta^{\#} = (a_1, a_3, a_5, a_7, a_9, a_{11}, a_{13}, a_2, a_4, a_6, a_8, a_{10}, a_{12})$.

Remark 1.37. (*i*) Let $\beta \in [13]$ of S_{13} where, $\beta_1 = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13})$

$\beta_1 = (a_1, a_3, a_5, a_7, a_9, a_{11}, a_{13}, a_2, a_4, a_6, a_8, a_{10}, a_{12})$
$\beta_2 = (a_1, a_4, a_7, a_{10}, a_{13}, a_3, a_6, a_9, a_{12}, a_2, a_5, a_8, a_{11})$
$\beta_3 = (a_1, a_5, a_9, a_{13}, a_4, a_8, a_{12}, a_3, a_7, a_{11}, a_2, a_6, a_{10})$
$\beta_4 = (a_1, a_6, a_{11}, a_3, a_8, a_{13}, a_5, a_{10}, a_2, a_7, a_{12}, a_4, a_9)$
$\beta_5 = (a_1, a_7, a_{13}, a_6, a_{12}, a_5, a_{11}, a_4, a_{10}, a_3, a_9, a_2, a_8)$

and d is a positive integer number we have:

(1)	$\beta^d = \beta \iff d \equiv 1 \pmod{13}$
(2)	$\beta^d = \beta_1 \iff d \equiv 2 \pmod{13}$
(3)	$\beta^d = \beta_2 \iff d \equiv 3 \pmod{13}$
(4)	$\beta^d = \beta_3 \iff d \equiv 4 \pmod{13}$
(5)	$\beta^d = \beta_4 \iff d \equiv 5 \pmod{13}$
(6)	$\beta^d = \beta_5 \iff d \equiv 6 \pmod{13}$
(7)	$\beta^d = \beta_5^{-1} \iff d \equiv 7 \pmod{13}$
(8)	$\beta^d = \beta_4^{-1} \iff d \equiv 8 \pmod{13}$
(9)	$\beta^d = \beta_3^{-1} \iff d \equiv 9 \pmod{13}$
(10)	$\beta^d = \beta_2^{-1} \iff d \equiv 10 \pmod{13}$
(11)	$\beta^d = \beta_1^{-1} \iff d \equiv 11 \pmod{13}$
(12)	$\beta^d = \beta^{-1} \iff d \equiv 12 \pmod{13}$

(*ii*) (1) $A(\beta) = A(\beta^{-1})$, $A(\beta_1) = A(\beta_1^{-1})$, $A(\beta_2) = A(\beta_2^{-1})$, $A(\beta_3) = A(\beta_3^{-1})$, $A(\beta_4) = A(\beta_4^{-1})$, and $A(\beta_5) = A(\beta_5^{-1})$ [Since for each $\lambda = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}) \in [13]^{\mp}$ in A_{13}, $\exists \mu = (b_1, b_{12})\ (b_2, b_{11})\ (b_3, b_{10})\ (b_4, b_9)\ (b_5, b_8)\ (b_6, b_7) \in A_9$ such that $\mu\lambda\mu^{-1} = \lambda^{-1}$].

(2) $A(\beta) = A(\beta_2)$ [Since $\exists t = (a_2, a_4, a_{10})\ (a_6, a_3, a_7)\ (a_{11}, a_5, a_{13})\ (a_8, a_9, a_{12}) \in A_{13}$ such that $t\beta t^{-1} = \beta_2$].

(3) $A(\beta) = A(\beta_3)$ [Since $\exists t = (a_6, a_8, a_3, a_9, a_7, a_{12})\ (a_{10}, a_{11}, a_2, a_5, a_4, a_{13}) \in A_{13}$ such that $t\beta t^{-1} = \beta_3$].

(4) $A(\beta_1) = A(\beta_4)$ [Since $\exists t = (a_7, a_3, a_6)\ (a_5, a_{11}, a_{13})\ (a_9, a_8, a_{12})\ (a_4, a_2, a_{10}) \in A_{13}$ such that $t\beta_1 t^{-1} = \beta_4$].

(5) $A(\beta_1) = A(\beta_5)$ [Since $\exists t = (a_2, a_4, a_{10})\ (a_6, a_3, a_7)\ (a_{11}, a_5, a_{13})\ (a_8, a_9, a_{12}) \in A_{13}$ such that $t\beta_1 t^{-1} = \beta_5$].

Theorem 1.38. *Let* $L = \{m \in N \mid m \equiv q (mod 13);$ for some $q = 1.3.4.9.10.12\}$. *If d is a positive integer number such that* $gcd(d, 13 = 1)$ *and* $\beta \in [13]$ *of* S_{13}, *then the solutions of* $x^d \in A(\beta)$ *in* A_{13} *are:*

(1) if $A(\beta)$ *if* $d \in L$.
(2) if $A^{\#}(\beta)$ *if* $d \notin L$.

Proof. Since $\beta \in [13] \cap H \cap A_{13}$, [13] splits into two classes $A(\beta)$ & $A^{\#}(\beta)$ of A_{13} and since $gcd(d, 13) = 1$, d, does not divide 13. Then by Taban (2007, lemma 3.2) we have $[13] = A(\beta) \cup A^{\#}(\beta)$ is a solution set of $x^d \in [13] = A(\beta) \cup A^{\#}(\beta)$ in S_{13}. However, $A(\beta) \cap A^{\#}(\beta) = \phi$, so for each $\pi \in [13] \Rightarrow (\pi \in A(\beta)\ \&\ \pi \notin A^{\#}(\beta))$ or $(\pi \notin A(\beta)\ \&\ \pi \in A^{\#}(\beta))$.

(1) Assume $d \in L$. If $\pi \in A(\beta)$, then we have $\pi \underset{A_{13}}{\approx} \beta$ (π conjugate to β in A_{13}). However, $\pi^d \underset{A_{13}}{\approx} \pi$ (since $d \in L$) $\Rightarrow \pi^d \underset{A_{13}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A^{\#}(\beta)$. If $\pi \in A^{\#}(\beta)$, we have $(\pi \underset{A_{13}}{\approx} \overset{\#}{\beta}$. But $\pi^d \underset{A_{13}}{\approx} \pi$ (since $(d \in L) \Rightarrow \pi^d \underset{A_{13}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A^{\#}(\beta)\ \&\ \pi^d \notin A(\beta)$. Then the solution set of $x^d \in A(\beta)$ in A_{13} is $A(\beta)$.

(2) Assume $d \notin L$, if $\pi \in A(\beta)$, then we have $\pi \underset{A_{13}}{\approx} \beta \Rightarrow \overset{\#}{\pi} \underset{A_{13}}{\approx} \overset{\#}{\beta}$. However, $\pi^d \underset{A_{13}}{\approx} \overset{\#}{\pi}$ (since $d \notin L$) $\Rightarrow \pi^d \underset{A_{13}}{\approx} \overset{\#}{\beta} \Rightarrow \pi^d \in A^{\#}(\beta) \& \pi^d \notin A(\beta)$. If $\pi \in A^{\#}(\beta)$, so $(\pi \underset{A_{13}}{\approx} \overset{\#}{\beta}) \Rightarrow \overset{\#}{\pi} \underset{A_{13}}{\approx} \beta$. But $\pi^d \underset{A_{13}}{\approx} \overset{\#}{\pi}$ (since $d \notin L$) $\Rightarrow \pi^d \underset{A_{13}}{\approx} \beta \Rightarrow \pi^d \in A(\beta) \& \pi^d \notin A^{\#}(\beta)$. Then the solution set of $x^d \in A(\beta)$ in A_{13} is $A^{\#}(\beta)$.

Lemma 1.39. *Let* $A(\beta)$ *be the conjugacy class of* β *in,* A_n, $n > 14$, *and* $\beta \in [k_1, k_2, \ldots, k_l] \cap H$, *where* $[k_1, k_2, \ldots, k_l] \in F_n$ *is a class of* S_n. *If p is a prime number such that* $gcd(p, k_i) = 1$, *for each i, then the solutions of* $x^p \in A(\beta)$ *are:*

(1) $[k_1, k_2, \ldots, k_l]^-$ *if* $\beta^p = (\beta^{-1}$ *or* $\gamma)$, *where* γ *is conjugate to* β^{-1}.
(2) $[k_1, k_2, \ldots, k_l]^+$ *if* $\beta^p = (\beta$ *or* $\gamma)$, *where* γ *is conjugate to* β.

Proof. $\beta \in A_n \cap H \cap [k_1, k_2, \ldots, k_l]$, $[k_1, k_2, \ldots, k_l]$, splits into two classes $[k_1, k_2, \ldots, k_l]^{\pm}$ of A_n, $[k_1, k_2, \ldots, k_l] \in F \Rightarrow A(\beta) = [k_1, k_2, \ldots, k_l]^+$, and $A(\beta^{-1}) = [k_1, k_2, \ldots, k_l]^-$. So, since $gcd(p, k_i) = 1$ for each i. Then by Taban (2007, lemma 2.8) we have for each $\lambda \in [k_1, k_2, \ldots, k_l] \Rightarrow \lambda^p \in [k_1, k_2, \ldots, k_l]$.

(1) Assume $\beta^p = (\beta^{-1}$ or $\lambda = b\beta^{-1}b^{-1}$; for some $b \in A_n)$, and let. $\lambda \in [k_1, k_2, \ldots, k_l]$. Then either $\lambda \in [k_1, k_2, \ldots, k_l]^+$ or $\lambda \in [k_1, k_2, \ldots, k_l]^-$.

If $\lambda \in [k_1, k_2, \ldots, k_l]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix},$$

$\lambda^p \in [k_1, k_2, \ldots, k_l]^- \Rightarrow \lambda^p \notin [k_1, k_2, \ldots, k_l]^+ = A(\beta)$,

If $\lambda \in [k_1, k_2, \ldots, k_l]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix},$$

$\lambda^p \in [k_1, k_2, \ldots, k_l]^+ = A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in A_n is $[k_1, k_2, \ldots, k_l]^-$.

(2) Assume $\beta^p = (\beta$ or$\lambda = b\beta b^{-1}$ for some $b \in A_n)$, and let $\lambda \in [k_1, k_2, \ldots, k_l]$. Then either $\lambda \in [k_1, k_2, \ldots, k_l]^+$ or $\lambda \in [k_1, k_2, \ldots, k_l]^-$.

If $\lambda \in [k_1, k_2, \ldots, k_l]^+$, $\exists t \in A_n \ni \lambda = t\beta t^{-1}$,

$$\lambda^p = t\beta^p t^{-1} = \begin{pmatrix} t\beta t^{-1} \\ \text{or} \\ tb\beta(tb)^{-1} \end{pmatrix},$$

$\lambda^p \in [k_1, k_2, \ldots, k_l]^+ = A(\beta) \Rightarrow \lambda^p \notin [k_1, k_2, \ldots, k_l]^-$,

If $\lambda \in [k_1, k_2, \ldots, k_l]^-$, $\exists t \in A_n \ni \lambda = t\beta^{-1}t^{-1}$,

$$\lambda^p = t\beta^{-p}t^{-1} = \begin{pmatrix} t\beta^{-1}t^{-1} \\ \text{or} \\ tb\beta^{-1}(tb)^{-1} \end{pmatrix},$$

$\lambda^p \in [k_1, k_2, \ldots, k_l]^- \Rightarrow \lambda^p \notin A(\beta)$.

Then the solution set of $x^p \in A(\beta)$ in A_n is $[k_1, k_2, \ldots, k_l]^+$.

Example 1.40. Let $\beta = \mu\lambda$, where $\mu = (1\ 2\ 3\ 4\ 5)$ and $\lambda = (6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16)$. Find the solutions of $x^{13} \in A(\beta)$ in A_{16}.

Solution:

since $n = 16 > 14$, $\beta \in [5,11] \cap H$, $[5,11] \in F_n$, $gcd(13,5) = 1$, $gcd(13,11) = 1$ and, $\exists \pi \in A_{16} \ni \pi\beta^{13}\pi^{-1} = \beta$ where $\beta^{13} = (1\ 4\ 2\ 5\ 3)(6\ 8\ 10\ 12\ 14\ 16\ 7\ 9\ 11\ 13\ 15)$ and $\pi = (3\ 5\ 4\ 2)(7\ 12\ 9\ 13\ 15\ 16\ 11\ 14\ 10\ 8)$. Then the solution set of $x^{13} \in A(\beta)$ in A_{16} is $[5,11]^+$.

Lemma 1.41. *Let $A(\beta)$ be the conjugacy class of β in A_n, $n > 14$, and, $\beta \in [k_1,k_2,\ldots,k_l] \cap H$ where $[k_1,k_2,\ldots,k_l] \in F_n$ is a class of. If p is a prime number such that $p \mid k_i$, for some i, then no solution of $x^p \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap H \cap [k_1,k_2,\ldots,k_l]$, $[k_1,k_2,\ldots,k_l]$, splits into two classes $[k_1,k_2,\ldots,k_l]^{\pm}$ of A_n and $[k_1,k_2,\ldots,k_l] \in F \Rightarrow A(\beta) = [k_1,k_2,\ldots,k_l]$. Also, since, then by Taban (2007, lemma 3.1) we have no solution for $x^p \in [k_i]$ in S_n. Then no solution for in S_n. So no solution for $x^p \in A(\beta)$ in A_n.

Lemma 1.42. *Let $A(\beta)$ be the conjugacy class of β in A_n, $n > 14$, and $\beta \in [k_1,k_2,\ldots,k_l] \cap H$, where $[k_1,k_2,\ldots,k_l] \in F_n$ is a class of S_n, p is a prime number, m is a positive integer. If for some $(1 \leqslant i \leqslant l)$ such that $p \mid k_i$, then no solution of $x^{p^m} \in A(\beta)$ in A_n.*

Proof. Since $\beta \in A_n \cap H \cap [k_1,k_2,\ldots,k_l]$, $[k_1,k_2,\ldots,k_l]$ splits into two classes $[k_1,k_2,\ldots,k_l]^{\pm}$ of A_n and $[k_1,k_2,\ldots,k_l] \in F \Rightarrow A(\beta) = [k_1,k_2,\ldots,k_l]^+$. Also, since $p \mid k_i$, then by Taban (2007, lemma 3.12) we have no solution for $x^{p^m} \in [k_i]$ in S_n. Then no solution for $x^{p^m} \in [k_1,k_2,\ldots,k_l]$ in S_n, so no solution for $x^{p^m} \in A(\beta)$ in A_n.

The number of solution

If β is an even permutation and $\beta \in C^{\alpha}(\beta) \cap H$, where $C^{\alpha}(\beta)$ is a class of β in S_n, we have $C^{\alpha}(\beta)$ splits into two classes $C^{\alpha}(\beta)^{\pm}$ of equal order $\Rightarrow A(\beta) = C^{\alpha}(\beta)^+$ or $C^{\alpha}(\beta)^-$, where $A(\beta)$ is a class of β in A_n. If $C^{\alpha}(\beta)^+$ or $C^{\alpha}(\beta)^-$ is a solution in A_n of any class equation in A_n, then the number of solutions is the number of all the elements that belong to the class $C^{\alpha}(\beta)^+$ or $C^{\alpha}(\beta)^-$. However, $|\ C^{\alpha}(\beta)^+\ | = |\ C^{\alpha}(\beta)^-\ | = \frac{|C^{\alpha}(\beta)|}{2}$. So the number of the solutions for the class equation $x^d = \beta$ in A_n is only $\frac{n!}{2z_{\alpha}}$.

Example 1.43. Find the solutions of $x^p \in A(234)$ in A_4 and the number of the solutions

(i) if $p = 13$.
(ii) if $p = 17$.

Solution:

$n = 4 \Rightarrow \beta = (2\ 3\ 4) \in [1,3]$. However, $[1,3] \subset H \Rightarrow \beta \in [1,3] \cap H$. Now we show that:

(i) If $p = 13$, then we have $gcd(13,3) = 1$, $gcd(13,1) = 1$, and $(2\ 3\ 4)^{13} = (2\ 3\ 4)$. Then by (1.12) the solution of $x^{13} \in A(2\ 3\ 4)$ in A_4 is $[1,3]^+$ and the number of the solution is $\frac{|[1,3]|}{2} = \frac{4!}{2\times 3} = 4$ permutations, where $[1,3]^- = \{(1\ 3\ 2),(2\ 3\ 4),(1\ 4\ 3),(1\ 2\ 4)\}$.

(ii) If $p = 17$, then we have $gcd(17,3) = 1$, $gcd(17,1) = 1$, and $(2\ 3\ 4)^{17} = (4\ 3\ 2) = (2\ 3\ 4)^{-1}$. Then by (1.12) the solution of $x^{17} \in \in A(2\ 3\ 4)$ in A_4 is $[1,3]^-$ and the number of the solution is $\frac{|[1,3]|}{2} = \frac{4!}{2\times 3} = 4$ permutations, where $[1,3]^- = \{(1\ 2\ 3),(2\ 4\ 3),(1\ 3\ 4),(1\ 4\ 2)\}$.

Example 1.44. Find the solutions of $x^{15} \in A(\beta)$ in A_7 and the number of the solutions where $\beta = (2\ 4\ 1\ 3\ 5\ 7\ 6)$.

Solution:

$n = 7 \Rightarrow \beta = (2\ 4\ 1\ 3\ 5\ 7\ 6) \in [7]$. However, $[7] \subset H \Rightarrow \beta \in [7] \cap H$. Assume $p = 3$, and $q = 5$, we have $gcd(3,7) = 1$ $gcd(5,7) = 1$, and $\beta^{15} = \beta$. Then by (1.15) the solution set of $x^{15} \in A(\beta)$ in A_7 is $[7]^+ = A(\beta)$ and the number of the solutions is $\frac{7!}{2\times 7} = 360$ permutations.

Example 1.45. Find the solutions of $x^{14} \in A((4\ 1\ 3)\ (2\ 6\ 7\ 5\ 8))$ in A_8 and the number of the solutions.

Solution:

$n = 8 \Rightarrow \beta = (4\ 1\ 3)(2\ 6\ 7\ 5\ 8) \in [3,5]$. However, $[3,5] \subset H \Rightarrow \beta \in [3,5] \cap H$. Let $p = 14$, then we have gcd(11,3) = 1, gcd(11,5) = 1, and $\beta^{14} = (1\,4\,3)(8\,5\,7\,6\,2) = \beta^{-1}$. Then by (1.12) the solution of $x^{14} \in A((4\ 1\ 3)(2\ 6\ 7\ 5\ 8))$ in A_8 is $[3,5]^-$ and the number of the solution is $\frac{8!}{2\times 3\times 5\times} = 1344$ permutations.

2. Conclusions

By the Cayley's theorem: Every finite group G is isomorphic to a subgroup of the symmetric group S_n, for some $n \geqslant 1$. Then we can discuss these propositions. Let $x^d = g$ be class equation in finite group G and assume that $f{:}G \cong A_n$, for some $n \notin \theta$ and $f(g) \in H \cap C^{\alpha}$. The first question we are concerned with is: what is the possible value of d provided that there is no solution for $x^d = g$ in G? The second question we are concerned with is: what is the possible value of d provided that there is a solution for $x^d = g$ in G? and then we can find the solution and the number of the solution for $x^d = g$ in G by using Cayley's theorem and our theorems in this paper. In another direction, let G be a finite group, and $\pi_i(G) = \{g \in G \mid i$ the least positive integer number satisfying $g^i = 1\}$. If $|\ \pi_i(G)\ | = k_i$, then we write $\pi_i(G) = \{g_{i1}, g_{i2}, \ldots, g_{ik_i}\}$, and $\prod = \{\pi_i(G)\}_{i\geqslant 1}$. For each $g \in G$ and $g_{ij} \in \pi_i(G)$ we have $(gg_{ij}g^{-1})^i = 1$. By the Cayley's theorem we can suppose that (f: $G \cong S_n$) or (f: $G \cong A_n$). Also the questions can be summarized as follows:

(1) Is $\prod = \{\pi_i(G)\}_{i\geqslant 1}$ collection set of conjugacy classes of G?
(2) Is there some $i \geqslant 1$, such that $f^{-1}(C^{\alpha}) = \pi_i(G)$, for each $C^{\alpha}$of A_n, where (f: $G \cong A_n$)?
(3) Is there some $i \geqslant 1$, such that $f^{-1}(C^{\alpha}) = \pi_i(G)$, for each $C^{\alpha}$of S_n, where (f: $G \cong S_n$)?
(4) If ($G \cong S_n$) and $p(n)$ is the number of partitions of n, is $|\ \prod\ | = p(n)$?
(5) If ($G \cong A_n$) and A_n has m ambivalent conjugacy classes. It is true that is also necessarily G has m ambivalent conjugacy classes?

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JAAS

Volume 10, September 2011

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Printed at UoB Printing Press